Contraction of Linear Differential and Difference Operators

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Krattenthaler's conjecture

Call $(c_n)_{n\geq 0}$ a *P-recursive sequence* over $\mathbb{Z}$ if

$$\ell_r c_n = \ell_{r-1} c_{n-1} + \cdots + \ell_0 c_{n-r}$$

where $\ell_i \in \mathbb{Z}[n]$, $\ell_r \neq 0$.

Conjecture: Let $(a_n)_{\geq 0}$ and $(b_n)_{\geq 0}$ be two P-recursive sequences over $\mathbb{Z}$ with leading coeff n. Show that $(n! a_n b_n)_{\geq 0}$ is also a P-recursive sequence over $\mathbb{Z}$ with leading coeff n.

Example for Krattenthaler's conjecture

Consider:

$$\begin{aligned} n a_n &= (31n-6)a_{n-1} + (49n-110)a_{n-2} + (9n-225)a_{n-3} \\ n b_n &= (4n+13)b_{n-1} + (69n-122)b_{n-2} + (36n-67)b_{n-3} \end{aligned}$$

$c_n := n! a_n b_n$ satisfies

$$\alpha(n) n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-9}$$

where $\alpha(n) \in \mathbb{Z}[n]$, $\deg_n(\alpha) = 20$.

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Known algorithms:

$$\beta n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-10}$$

where β is a 853-digit integer.

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Our algorithm:

$$1 n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-14}$$

Ore algebra (shift case)

$$\mathbb{Z}[n][\partial] \quad \subset \quad \mathbb{Q}(n)[\partial]$$

small ring big ring

Assume $L = \ell_r \partial^r + \cdots + \ell_1 \partial + \ell_0 \in \mathbb{Z}[n][\partial]$. Then

$$L \circ f(n) = \ell_r f(n+r) + \cdots + \ell_1 f(n+1) + \ell_0 f(n)$$

- Call L an annihilator of f if $L \circ f = 0$.
- Call $\deg_\partial(L) := r$ the order of L, $\mathrm{lc}_\partial(L) := \ell_r$ the leading coeff
- Let $T \in \mathbb{Z}[n][\partial]$. Call T a left multiple of L if $T = PL$, where $P \in \mathbb{Q}(n)[\partial]$.

Certifying integer sequences

Example 1 Consider an annihilator of $u(n)$:

$$L = (1+16n)^2\partial^2 - (224+512n)\partial - (1+n)(17+16n)^2$$

Question: Assume $u(0), u(1) \in \mathbb{Z}$, whether or not $u(n) \in \mathbb{Z}$ for each $n \in \mathbb{N}$?

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(Abramov, Barkatou, van Hoeij, 2006):

$$T := (\ldots)L = 64\partial^3 + \text{ lower terms } \in \mathbb{Z}[n][\partial]$$

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Our algorithm:

$$\widetilde{T} := 1\partial^3 + \text{ lower terms } \in \mathbb{Z}[n][\partial]$$

Answer: Yes, $u(n)$ is an integer sequence.

Desingularization

Given $L \in \mathbb{Z}[n][\partial]$, $p \mid \mathrm{lc}_\partial(L)$.

Let $T \in \mathbb{Z}[n][\partial]$ with $\mathrm{lc}_\partial(T) = a \cdot g$, $a \in \mathbb{Z}$, g primitive.
Call T a p-removed operator of L if

- T is a left multiple of L
- $g \mid \frac{1}{p} \mathrm{lc}_\partial(L)$

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Note: a is called the content of $\mathrm{lc}_\partial(T)$, denoted as $\mathrm{c}(T)$.

Desingularization

Let T be a p-removing operator.

- Call T a desingularized operator of L if

$$\deg(\mathrm{lc}_\partial(T)) = \min\{\deg(\mathrm{lc}_\partial(Q)) \mid Q \text{ is a p-removed operator}\}$$

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- If T is a desingularized operator and

  $$\mathrm{c}(T) = \min\{\mathrm{c}(Q) \mid Q \text{ is a desingularized operator}\},$$

  call T a completely desingularized operator of L.

Desingularization

Example 1 (continued) Consider:

$$L = (1 + 16n)^2 \partial^2 - (224 + 512n)\partial - (1 + n)(17 + 16n)^2$$

(Abramov et al. 2006):

$$T = (\dots)L = 64\partial^3 + \text{ lower terms } \in \mathbb{Z}[n][\partial]$$

Our algorithm:

$$\widetilde{T} = 1\partial^3 + \text{ lower terms } \in \mathbb{Z}[n][\partial]$$

T and $\widetilde{T}$ are desingularized and completely desingularized operators, resp.

Contraction

Given $L \in \mathbb{Z}[n][\partial]$, let $\langle L \rangle := \mathbb{Q}(n)[\partial]L$.

The contraction ideal of $\langle L \rangle$ is

$$\mathrm{Cont}(L) := \langle L \rangle \cap \mathbb{Z}[n][\partial]$$

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The contraction ideal of $\langle L \rangle$ is

$$\mathrm{Cont}(L) := \langle L \rangle \cap \mathbb{Z}[n][\partial]$$

- $\mathrm{Cont}(L)$ is finitely generated.
- Every desingularized operator of L belongs to $\mathrm{Cont}(L)$.
- $\mathrm{Cont}(L)$ contains $\mathbb{Z}[n][\partial]L$, but in general more operators.

Contraction

Goal: compute a $\mathbb{Z}[n][\partial]$-basis of $\mathrm{Cont}(L)$.

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Example 1 (continued) Consider:

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$\mathrm{Cont}(L)$ is generated by $\{L, \widetilde{T}\}$.

Desingularization and contraction

Let $L \in \mathbb{Z}[n][\partial]$ with $\deg_{\partial}(L) = r$.

Set $k \geq r$. Call

$$M_k := \{ T \mid T \in \mathrm{Cont}(L),\ \deg_{\partial}(T) \leq k \}$$

the k-th submodule of $\mathrm{Cont}(L)$.

Desingularization and contraction

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the k-th submodule of $\mathrm{Cont}(L)$.

Theorem 1 (Main Result 1) Let T be a desingularized operator of L. If $k = \deg_\partial(T)$, then

$$\mathrm{Cont}(L) = (\mathbb{Z}[x][\partial] \cdot M_k) : \mathrm{c}(T)^\infty$$

Order bound for desingularized operators

Let $L \in \mathbb{Z}[n][\partial]$.

Assume $p \mid \mathrm{lc}_\partial(L)$, p is irreducible.

- If p is removable, then one can compute an integer k, s.t. there exists a p-removing operator of order k.
- Using Euclidean algorithm, one can compute an order bound for desingularized operators.

Chen, Jaroschek, Kauers, Singer. Desingularization explains order-degree curves for Ore operators. *ISSAC 2013*.

Determining the k-th submodule

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$.

Question: Given $k \geq r$, compute a $\mathbb{Z}[n]$-spanning set of M_k?

Determining the k-th submodule

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Question: Given $k \geq r$, compute a $\mathbb{Z}[n]$-spanning set of M_k?

1. Make an ansatz: $F = z_k \partial^k + \ldots + z_0$, where $z_k, \ldots, z_0 \in \mathbb{Z}[n]$ are to be determined.
2. Compute $\text{rrem}(F, L) = 0$. It gives:

$$(z_k, \ldots, z_0)A = \mathbf{0}, \tag{1}$$

   where $A \in \mathbb{Z}[n]^{(k+1)\times r}$.
3. Using Gröbner bases, solve (1).

Computing desingularized operators

Let $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

Question: Assume k is an order bound for desingularized operators, compute a desingularized operator?

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Set $k \geq r$. Call

$$I_k := \left\{ [\partial^k]P \mid P \in M_k \right\} \cup \{0\},$$

the k-th coefficient ideal of $\mathrm{Cont}(L)$, where $[\partial^k]P$ is the coefficient of ∂^k in P.

Computing desingularized operators

Proposition If $\{B_1, \ldots, B_t\}$ is a spanning set of M_k, then

$$I_k = \langle [\partial^k] B_1, \ldots, [\partial^k] B_t \rangle$$

Computing desingularized operators

Proposition If $\{B_1, \ldots, B_t\}$ is a spanning set of M_k, then

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Theorem 3 If s is a nonzero element of I_k with minimal degree, then S in M_k with $\mathrm{lc}_\partial(S) = s$ is a desingularized operator.

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Theorem 3 If s is a nonzero element of I_k with minimal degree, then S in M_k with $\mathrm{lc}_\partial(S) = s$ is a desingularized operator.

Note: Using Euclidean algorithm over $\mathbb{Q}[n]$, one can compute an operator S with $\mathrm{lc}_\partial(S) = s$.

Determining contraction ideals

Algorithm 1: Given $L \in \mathbb{Z}[n][\partial]$, compute a basis of $\mathrm{Cont}(L)$.

1. Compute an order bound k for desingularized operators.
2. Compute a spanning set of M_k.
3. Compute a desingularized operator T of order k.
4. Using Gröbner bases, compute a basis of

$$(\mathbb{Z}[n][\partial] \cdot M_k) : \mathrm{c}(T)^\infty.$$

Determining contraction ideals

Example 1 (continued) Consider:

$$L = (1 + 16n)^2\partial^2 - (224 + 512n)\partial - (1 + n)(17 + 16n)^2$$

1. An order bound for desingularized operator is 3.
2. M_3 is generated by $\{L, \widetilde{T}\}$.
3. Since $\mathrm{lc}_\partial(\widetilde{T}) = 1$, $\widetilde{T}$ is a desingularized operator.
4. $\mathrm{Cont}(L) = (\mathbb{Z}[n][\partial] \cdot \{L, \widetilde{T}\}) : 1^\infty = \mathbb{Z}[n][\partial] \cdot \{L, \widetilde{T}\}$.

Computing completely desingularized operators

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

Recall: Let T be a desingularized operator.
Call T a completely desingularized operator of L if

$$c(T) = \min\{c(Q) \mid Q \text{ is a desingularized operator}\}$$

Computing completely desingularized operators

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$.

Recall: Let T be a desingularized operator.
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Question: compute a completely desingularized operator of L?

Main result 2

Theorem 4 Assume $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$ and **G** is a Gröbner basis of I_k. Let f be the element of **G** with minimal degree. If $F \in \mathrm{Cont}(L)$ with $\mathrm{lc}_\partial(F) = f$, then F is a completely desingularized operator of L.

Main result 2

Theorem 4 Assume $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$ and **G** is a Gröbner basis of I_k. Let f be the element of **G** with minimal degree. If $F \in \mathrm{Cont}(L)$ with $\mathrm{lc}_\partial(F) = f$, then F is a completely desingularized operator of L.

Algorithm 2: Given $L \in \mathbb{Z}[n][\partial]$, compute a completely desingularized operator of L.

1. By Algorithm 1, $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$.
2. Compute a Gröbner basis **G** of I_k.
3. Let f be the element of **G** with minimal degree. Tracing back to step 2, find $F \in \mathrm{Cont}(L)$ with $\mathrm{lc}_\partial(F) = f$.

Example 2

Consider:

$$\begin{aligned} na_n &= (31n-6)a_{n-1} + (49n-110)a_{n-2} + (9n-225)a_{n-3} \\ nb_n &= (4n+13)b_{n-1} + (69n-122)b_{n-2} + (36n-67)b_{n-3} \end{aligned}$$

$c_n := n!a_nb_n$ has an annihilator L of order 9
with $\mathrm{lc}_\partial(L) = (n+9)\alpha(n)$, $\alpha(n) \in \mathbb{Z}[n]$.

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1. By Algorithm 1, $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_{14}$
2. $I_{14} = \langle n+14 \rangle$
3. Find a completely desingularized operator T of L,
   $\mathrm{lc}_\partial(T) = n + 14$

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Translating T into a recurrence equation of c_n

$$1nc_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-14}$$

Krattenthaler's conjecture

Let $(a_n)_{\geq 0}$ and $(b_n)_{\geq 0}$ be two P-recursive sequences over $\mathbb{Z}$ with leading coeff n.

Set $L \in \mathbb{Z}[n][\partial]$ to be an annihilator of $n! a_n b_n$, and T to be a completely desingularized operator.

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Set $L \in \mathbb{Z}[n][\partial]$ to be an annihilator of $n! a_n b_n$, and T to be a completely desingularized operator.

Then

$$\text{Krattenthaler's conjecture holds}$$

$$\Updownarrow$$

$$\text{lc}_\partial(T) = n + \deg_\partial(T)$$

Case 1

Consider:

$$\begin{aligned} na_n &= \alpha a_{n-1} \\ nb_n &= \beta_1 b_{n-1} + \ldots + \beta_t b_{n-t} \end{aligned}$$

with α, $\beta_i \in \mathbb{Z}[n]$. Then $c_n := n! a_n b_n$ satisfies

$$nc_n = \gamma_1 c_{n-1} + \ldots + \gamma_t c_{n-t}$$

where $\gamma_i := \beta_i \prod_{j=0}^{i-1} \alpha(n-j)$.

Case 2

Consider:

$$
\begin{aligned}
na_n &= \alpha_1 a_{n-1} + \alpha_2 a_{n-2} \\
nb_n &= \beta_1 b_{n-1} + \beta_2 b_{n-2} + \beta_3 b_{n-3}
\end{aligned}
$$

where α_i, β_j are indeterminates. Then $c_n := n! a_n b_n$ satisfies

$$
nc_n = \gamma_1 c_{n-1} + \ldots + \gamma_9 c_{n-9}
$$

with $\gamma_i \in \mathbb{Z}[\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, n]$.

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- An algorithm for computing completely desingularized operators
- Certify integer sequences and check special cases of Krattenthaler's conjecture.

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Remark: Using Tsai's bound, Algorithm 1 can determine contraction of a differential operator.

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- An algorithm for computing completely desingularized operators
- Certify integer sequences and check special cases of Krattenthaler's conjecture.

Remark: Using Tsai's bound, Algorithm 1 can determine contraction of a differential operator.

Thanks!