Contraction of Ore Ideals with Applications

Yi Zhang*
Institute for Algebra, Johannes Kepler University, Linz A-4040, Austria
KLMM, AMSS, Chinese Academy of Sciences, Beijing 100190, China
zhangy@amss.ac.cn

ABSTRACT

Ore operators form a common algebraic abstraction of linear ordinary differential and recurrence equations. Given an Ore operator L with polynomial coefficients in x, it generates a left ideal I in the Ore algebra over the field $\mathbf{k}(x)$ of rational functions. We present an algorithm for computing a basis of the contraction ideal of I in the Ore algebra over the ring $R[x]$ of polynomials, where R may be either $\mathbf{k}$ or a domain with $\mathbf{k}$ as its fraction field. This algorithm is based on recent work on desingularization for Ore operators by Chen, Jaroschek, Kauers and Singer. Using a basis of the contraction ideal, we compute a completely desingularized operator for L whose leading coefficient not only has minimal degree in x but also has minimal content. Completely desingularized operators have interesting applications such as certifying integer sequences and checking special cases of a conjecture of Krattenthaler.

Keywords

Ore Algebra, Desingularization, Contraction, Syzygy

1. INTRODUCTION

There are various reasons why linear differential equations are easier than non-linear ones. One is of course that the solutions of linear differential equations form a vector space over the underlying field of constants. Another important feature concerns the singularities. While for a nonlinear differential equation the location of the singularity may depend continuously on the initial value, this is not possible for linear equations. Instead, a solution f of a differential equation

$$a_0(x)f(x) + \cdots + a_r(x)f^{(r)}(x) = 0,$$

where $a_0, \ldots, a_r$ are some analytic functions, can only have singularities at points $\xi \in \mathbb{C}$ with $a_r(\xi) = 0$.

*Supported by the Austrian Science Fund (FWF) grants Y464-N18, NSFC grants (91118001, 60821002/F02) and a 973 project (2011CB302401).

ISSAC'16, ISSAC '16, July 19 - 22, 2016, Waterloo, ON, Canada

ACM 978-1-4503-4380-0/16/07 ... $15.00.
DOI: http://dx.doi.org/10.1145/2930889.2930890

In this article, we consider the case where $a_0, \ldots, a_r$ are polynomials. In this case, a_r can have only finitely many roots. We shall also consider the case of recurrence equations

$$a_0(n)f(n) + \cdots + a_r(n)f(n+r) = 0,$$

where again there is a strong connection between the roots of a_r and the singularities of a solution.

While every singularity of a solution leaves a trace in the leading coefficient of an equation, the converse is not true. In general, the leading coefficient a_r may have roots at a point where no solution is singular. Such points are called apparent singularities, and it is sometimes of interest to identify them. One technique for doing so is called desingularization. As an example, consider the recurrence operator

$$L = (1 + 16n)^2\partial^2 - 32(7 + 16n)\partial - (1 + n)(17 + 16n)^2,$$

which is taken from [1, Section 4.1]. Here, ∂ denotes the shift operator $f(n) \mapsto f(n+1)$. For any choice of two initial values $u_0, u_1 \in \mathbb{Q}$, there is a unique sequence $u\colon \mathbb{N} \to \mathbb{Q}$ with $u(0) = u_0$, $u(1) = u_1$ and L applied to u gives the zero sequence. A priori, it is not obvious whether or not u is actually an integer sequence, if we choose u_0, u_1 from $\mathbb{Z}$, because the calculation of the $(n+2)$nd term from the earlier terms via the recurrence encoded by L requires a division by $(1 + 16n)^2$, which could introduce fractions. In order to show that this division never introduces a denominator, the authors of [1] note that every solution of L is also a solution of its left multiple

$$\begin{aligned} T &= \left(\frac{64}{(17+16n)^2}\partial + \frac{(23+16n)(25+16n)}{(17+16n)^2}\right)L \\ &= 64\partial^3 + (16n+23)(16n-7)\partial^2 - (576n+928)\partial \\ &\quad -(16n+23)(16n+25)(n+1). \end{aligned}$$

The operator T has the interesting property that the factor $(1 + 16n)^2$ has been "removed" from the leading coefficient. This is, however, not quite enough to complete the proof, because now a denominator could still arise from the division by 64 at each calculation of a new term via T. To complete the proof, the authors show that the potential denominators introduced by $(1 + 16n)^2$ and by 64, respectively, are in conflict with each other, and therefore no such denominators can occur at all.

The process of obtaining the operator T from L is called desingularization, because there is a polynomial factor in the leading coefficient of L which does not appear in the leading coefficient of T. In the example above, the price to be paid for the desingularization was a new constant factor 64 which appears in the leading coefficient of T but not in the original leading coefficient of L. Desingularization algorithms in the literature [2, 1, 3, 7, 8] care only about the removal of polynomial factors without introducing new polynomial factors, but they do not consider the possible introduction of new constant factors. A contribution of the present paper is a desingularization algorithm which minimizes, in a sense, also any constant factors introduced during the desingularization. For example, for the operator L above, our algorithm finds the alternative desingularization

$$\begin{aligned}\tilde{T} \quad = \quad & \partial^3 + \left(128n^3 - 104n^2 - 11n - 3\right)\partial^2 + \\ & \left(-256n^2 + 127n + 94\right)\partial - \\ & (128n^2 + 24n - 131)(1+n)^2,\end{aligned} \tag{1}$$

which immediately certifies the integrality of its solutions.

In more algebraic terms, we consider the following problem. Given an operator $L \in \mathbb{Z}[x][\partial]$, where $\mathbb{Z}[x][\partial]$ is an Ore algebra (see Section 2 for definitions), we consider the left ideal $\langle L\rangle = \mathbb{Q}(x)[\partial]L$ generated by L in the extended algebra $\mathbb{Q}(x)[\partial]$. The contraction of $\langle L\rangle$ to $\mathbb{Z}[x][\partial]$ is defined as $\mathrm{Cont}(L) := \langle L\rangle \cap \mathbb{Z}[x][\partial]$. This is a left ideal of $\mathbb{Z}[x][\partial]$ which contains $\mathbb{Z}[x][\partial]L$, but in general more operators. Our goal is to compute a $\mathbb{Z}[x][\partial]$-basis of $\mathrm{Cont}(L)$. In the example above, such a basis is given by $\{L, \tilde{T}\}$ (see Example 4.8). The traditional desingularization problem corresponds to computing a basis of the $\mathbb{Q}[x][\partial]$-left ideal $\langle L\rangle \cap \mathbb{Q}[x][\partial]$.

The contraction problem for Ore algebras $\mathbb{Q}[x][\partial]$ was proposed by Chyzak and Salvy [10, Section 4.3]. For the analogous problem in commutative polynomial rings, there is a standard solution via Gröbner bases [4, Section 8.7]. It reduces the contraction problem to a saturation problem. This reduction also works for the differential case, but in that case it is not so helpful because it is less obvious how to solve the saturation problem. A solution was proposed by Tsai [24], which involves homological algebra and D-modules theory. Our work is based on desingularization for Ore operators in [7, 8]. In particular, the p-removing operator in [8, Lemma 4] provides us with a key to determine contraction ideals. The algorithm developed in this paper applies to both differential and difference cases. Moreover, we compute a completely desingularized operator in a contraction ideal, which has minimal leading coefficient in terms of both degree and content.

The rest of this paper is organized as follows. In Section 2, we describe Ore polynomial rings over a principal ideal domain, and extend the notion of p-removed operators to them. The notion of desingularized operators is defined and connected with contraction ideals in Section 3. We determine a contraction ideal in Section 4, and compute completely desingularized operators in Section 5.

2. PRELIMINARIES

This section is divided into three parts. First, we describe Ore algebras that are used in the paper. Second, we extend the notion of p-removed operators in [7, 8]. At last, we make some remarks on Gröbner basis computation over a principal ideal domain.

2.1 Ore Algebra

Throughout the paper, we let R be a principal ideal domain. For instance, R can be the ring of integers or that of univariate polynomials over a field. Note that $R[x]$ is a unique factorization domain. So every nonzero polynomial f in $R[x]$ can be written as cg, where $c \in R$ and $g \in R[x]$ whose coefficients have the trivial greatest common divisor. We call c the content and g the primitive part of f. They are unique up to the units in R.

We consider the *Ore algebra* $R[x][\partial;\sigma,\delta]$, where $\sigma : R[x] \to R[x]$ is a ring automorphism that leaves the elements of R fixed, and $\delta : R[x] \to R[x]$ is a σ-derivation, *i.e.*, an R-linear map satisfying the skew Leibniz rule

$$\delta(fg) = \sigma(f)\delta(g) + \delta(f)g \quad \text{for } f, g \in R[x].$$

The addition in $R[x][\partial;\sigma,\delta]$ is coefficient-wise and the multiplication is defined by associativity via the commutation rule $\partial p = \sigma(p)\partial + \delta(p)$ for $p \in R[x]$. The ring $R[x][\partial;\sigma,\delta]$ is abbreviated as $R[x][\partial]$ when σ and δ are clear from the context.

Given $L \in R[x][\partial]$, we can uniquely write it as

$$L = \ell_r\partial^r + \ell_{r-1}\partial^{r-1} + \cdots + \ell_0$$

with $\ell_0, \ldots, \ell_r \in R[x]$ and $\ell_r \neq 0$. We call r the *order* and ℓ_r the *leading coefficient* of L. They are denoted by $\deg_\partial(L)$ and $\mathrm{lc}_\partial(L)$, respectively. Assume that $P \in R[x][\partial]$ is of order k. A repeated use of the commutation rule yields

$$\mathrm{lc}_\partial(PL) = \mathrm{lc}_\partial(P)\sigma^k(\mathrm{lc}_\partial(L)). \tag{2}$$

For a subset S of $R[x][\partial]$, the left ideal generated by S is denoted by $R[x][\partial]\cdot S$.

Let Q_R be the quotient field of R. Then $Q_R(x)[\partial]$ is an Ore algebra containing $R[x][\partial]$. For $L \in R[x][\partial]$, we define the *contraction ideal* of L to be $Q_R(x)[\partial]L \cap R[x][\partial]$ and denote it by $\mathrm{Cont}(L)$.

2.2 Removability

We generalize some terminologies given in [7, 8] by replacing the coefficient ring $\mathbb{K}[x]$ with $R[x]$, where $\mathbb{K}$ is a field.

Definition 2.1. *Let $L \in R[x][\partial]$ with positive order, and p be a divisor of $\mathrm{lc}_\partial(L)$ in $R[x]$.*

(i) We say that p is removable *from L at order k if there exist $P \in Q_R(x)[\partial]$ with order k, and $w, v \in R[x]$ with $\gcd(p, w) = 1$ in $R[x]$ such that*

$$PL \in R[x][\partial] \quad \text{and} \quad \sigma^{-k}(\mathrm{lc}_\partial(PL)) = \frac{w}{vp}\,\mathrm{lc}_\partial(L).$$

We call P a p-removing operator for L over $R[x]$*, and PL the corresponding* p-removed operator.

(ii) p is simply called removable *from L if it is removable at order k for some $k \in \mathbb{N}$. Otherwise, p is called* non-removable *from L.*

Note that every p-removed operator lies in $\mathrm{Cont}(L)$.

Example 2.2. *In the example of Section 1, $(1 + 16n)^2$ is removable from L at order 1. And T is the corresponding $(1 + 16n)^2$-removed operator for L.*

Example 2.3. *In the differential Ore algebra $\mathbb{Z}[x][\partial]$, where $\partial x = x\partial + 1$, let $L = x(x-1)\partial - 1$. Then $(1-x)\partial^2 - 2\partial = \left(\frac{1}{x}\partial\right)L$ is an x-removed operator for L (see [8, Example 3]).*

The authors of [7] provide a convenient form of p-removing operators over $\mathbb{K}[x]$ in order to get an order bound. We derive a similar form over $R[x]$ and use it in Section 5.

Lemma 2.4. *Let $L \in R[x][\partial]$ with positive order. Assume that $p \in R[x]$ is removable from L at order k. Then there exists a p-removing operator for L over $R[x]$ in the form*

$$\frac{p_0}{\sigma^k(p)^{d_0}} + \frac{p_1}{\sigma^k(p)^{d_1}}\partial + \cdots + \frac{p_k}{\sigma^k(p)^{d_k}}\partial^k,$$

where p_i belongs to $R[x]$, $\gcd(p_i, \sigma^k(p)) = 1$ in $R[x]$ or $p_i = 0$, $i = 0, 1, \ldots, k$, and $d_k \geq 1$.

Proof. By (2) and Definition 2.1 (i), $\mathrm{lc}_\partial(P) = \sigma^k(w/(vp))$ for some w, v in $R[x]$ with $\gcd(w, p) = 1$. Then we can write a p-removing operator for L over $R[x]$ in the form

$$P = \frac{p_0}{q_0\sigma^k(p)^{d_0}} + \frac{p_1}{q_1\sigma^k(p)^{d_1}}\partial + \cdots + \frac{p_k}{q_k\sigma^k(p)^{d_k}}\partial^k,$$

where $p_i, q_i \in R[x]$, $\gcd(p_iq_i, \sigma^k(p)) = 1$ in $R[x]$ or $p_iq_i = 0$, $i = 0, \ldots, k$, $d_k \geq 1$. Let $\tilde{P} = \left(\prod_{i=0}^k q_i\right) P$, $\tilde{p}_i = p_i \left(\prod_{j=0}^k q_j\right)/q_i$, $i = 0, \ldots, k$. Then

$$\tilde{P} = \frac{\tilde{p}_0}{\sigma^k(p)^{d_0}} + \frac{\tilde{p}_1}{\sigma^k(p)^{d_1}}\partial + \cdots + \frac{\tilde{p}_k}{\sigma^k(p)^{d_k}}\partial^k,$$

where $\gcd(\tilde{p}_i, \sigma^k(p)) = 1$ in $R[x]$ or $\tilde{p}_i = 0$, $i = 0, \ldots, k$. Moreover,

$$\sigma^{-k}(\mathrm{lc}_\partial(\tilde{P}L)) = \frac{\sigma^{-k}(\tilde{p}_k)}{p^{d_k}}\,\mathrm{lc}_\partial(L).$$

By Definition 2.1, $\tilde{P}$ is a p-removing operator for L over $R[x]$ with the required form. □

2.3 Gröbner bases

In Sections 4 and 5, we will make essential use of Gröbner bases in $R[x][\partial]$. When $R = \mathbf{k}[t]$ with $\mathbf{k}$ being a field, the notion of Gröbner bases and Buchberger's algorithm are available [15]. Furthermore, the corresponding implementation is available in [16]. In our case, σ is an R-automorphism of $R[x]$, which implies that $\sigma(x) = ax + b$ where a, b are in R and a is a unit. Let $\prec$ be a term order on $\{x^i\partial^j \mid i, j \in \mathbb{N}\}$. For any non-zero operator $P \in R[x][\partial]$, we define the *head term* of P to be the highest term appearing in P with respect to $\prec$, and denote it by $\mathrm{HT}(P)$. The coefficient of $\mathrm{HT}(P)$ is called the *head coefficient* of P. Let c be the head coefficient of P with respect to $\prec$. By the commutation rule, $\partial^i P$ has head coefficient ca^i, which is associated to c, because a^i is a unit. This observation enables us to extend the notion of Gröbner bases and Buchberger's algorithm in [4, 22] to Ore case in a straightforward way. For details, see [18, 6] and [20, IV.46.13].

3. DESINGULARIZATION AND CONTRACTION

In this section, we define the notion of desingularized operators, and connect it with contraction ideals. As a matter of notation, for an operator $L \in R[x][\partial]$, we set

$$M_k(L) = \{P \in \mathrm{Cont}(L) \mid \deg_\partial(P) \leq k\}.$$

Note that $M_k(L)$ is a left $R[x]-$submodule of $\mathrm{Cont}(L)$. We call it the kth submodule of $\mathrm{Cont}(L)$. When the operator L is clear from context, $M_k(L)$ is simply denoted by M_k.

Definition 3.1. *Let $L \in R[x][\partial]$ with order $r > 0$, and*

$$\mathrm{lc}_\partial(L) = cp_1^{e_1}\cdots p_m^{e_m}, \tag{3}$$

where $c \in R$ and $p_1, \ldots, p_m \in R[x] \setminus R$ are irreducible and pairwise coprime. An operator $T \in R[x][\partial]$ of order k is called a desingularized operator for L *if $T \in \mathrm{Cont}(L)$ and*

$$\sigma^{r-k}(\mathrm{lc}_\partial(T)) = \frac{a}{bp_1^{k_1}\cdots p_m^{k_m}}\,\mathrm{lc}_\partial(L), \tag{4}$$

where $a, b \in R$ with $b \neq 0$, and $p_i^{d_i}$ is non-removable from L for each $d_i > k_i$, $i = 1 \ldots m$.

Desingularized operators always exist by [8, Lemma 4].

Lemma 3.2. *Let $L \in R[x][\partial]$ be of order $r > 0$, and $k \in \mathbb{N}$ with $k \geq r$. Assume that T is a desingularized operator for L and $\deg_\partial(T) = k$.*

(i) $\deg_x(\mathrm{lc}_\partial(T)) = \min\{\deg_x(\mathrm{lc}_\partial(Q)) \mid Q \in M_k(L) \setminus \{0\}\}$.

(ii) $\partial^i T$ is a desingularized operator for L for each $i \in \mathbb{N}$.

(iii) Set $\mathrm{lc}_\partial(T) = ag$, where $a \in R$ and $g \in R[x]$ is primitive. Then, for all $F \in \mathrm{Cont}(L)$ of order j with $j \geq k$, $\sigma^{j-k}(g)$ divides $\mathrm{lc}_\partial(F)$ in $R[x]$.

Proof. (i) Let $t = \mathrm{lc}_\partial(T)$ and

$$d = \min\{\deg_x(\mathrm{lc}_\partial(Q)) \mid Q \in M_k(L) \setminus \{0\}\}.$$

Suppose that $d < \deg_x(t)$. Then there exists $Q \in \mathrm{Cont}(L)$ with $\deg_x(\mathrm{lc}_\partial(Q)) = d$. Without loss of generality, we can assume that $\deg_\partial(Q) = k$, because the leading coefficients of Q and $\partial^i Q$ are of the same degree for all $i \in \mathbb{N}$.

By pseudo-division in $R[x]$, we have that

$$st = q\,\mathrm{lc}_\partial(Q) + h$$

for some $s \in R\setminus\{0\}$, $q, h \in R[x]$, and $h = 0$ or $\deg_x(h) < d$. If h were nonzero, then $sT - qQ$ would be a nonzero operator of order k in $\mathrm{Cont}(L)$ whose leading coefficient is of degree less than d, a contradiction. Thus, $st = q\,\mathrm{lc}_\partial(Q)$. In particular, $\deg_x(q)$ is positive, as $d < \deg_x(t)$. It follows from (4) that

$$\sigma^{r-k}(\mathrm{lc}_\partial(Q)) = \sigma^{r-k}\left(\frac{st}{q}\right) = \frac{sa}{\sigma^{r-k}(q)bp_1^{k_1}\cdots p_m^{k_m}}\,\mathrm{lc}_\partial(L),$$

which belongs to $R[x]$. Hence, $\sigma^{r-k}(q)$ divides $\mathrm{lc}_\partial(L)$ in $R[x]$. Consequently, there exists $i \in \{1 \ldots m\}$ such that p_i divides $\sigma^{r-k}(q)$ in $R[x]$. This implies that p^{k_i+1} is removable from L, a contradiction.

(ii) It is immediate from Definition 3.1.

(iii) Let $\mathrm{lc}_\partial(F) = uf$, where $u \in R$ and f is primitive in $R[x]$. By (ii), $\partial^{j-k}T$ is a desingularized operator whose leading coefficient equals $a\sigma^{j-k}(g)$. A similar argument used in the proof of the first assertion implies that

$$vf = p\sigma^{j-k}(g) \quad \text{for some } v \in R \setminus \{0\} \text{ and } p \in R[x].$$

By Gauss's Lemma in $R[x]$, $\sigma^{j-k}(g) \mid f$. □

We describe a relation between desingularized operators and contraction ideals. Let I be a left ideal in $R[x][\partial]$, and $a \in R$. The saturation of I with respect to a is defined to be

$$I : a^\infty = \left\{P \in R[x][\partial] \mid a^i P \in I \text{ for some } i \in \mathbb{N}\right\}.$$

Since a is a constant with respect to σ and δ, the saturation $I : a^\infty$ is a left ideal.

Theorem 3.3. *Let $L \in R[x][\partial]$ with order $r > 0$. Assume that T is a desingularized operator for L. Let $\mathrm{lc}_\partial(T)=ag$, where $a \in R$ and g is primitive in $R[x]$. If k is such that $T \in M_k$ for some $k \in \mathbb{N}$, then*

$$\mathrm{Cont}(L) = (R[x][\partial] \cdot M_k) : a^\infty.$$

Proof. By Lemma 3.2 (ii), we may assume that the order of T is equal to k. Set $J = (R[x][\partial] \cdot M_k) : a^\infty$.

First, assume that $F \in J$. Then there exists $j \in \mathbb{N}$ such that $a^j F \in R[x][\partial] \cdot M_k$. It follows that $F \in Q_R(x)[\partial]L$. Thus, $F \in \mathrm{Cont}(L)$ by definition.

Next, note that $\mathrm{Cont}(L) = \cup_{i=r}^{\infty} M_i$ and that $M_i \subseteq M_{i+1}$. It suffices to show $M_i \subseteq J$ for all $i \geq k$. We proceed by induction on i.

For $i = k$. $M_k \subseteq J$ by definition.

Suppose that the claim holds for i. For any $F \in M_{i+1} \setminus M_i$, $\deg_\partial(F) = i+1$. By Lemma 3.2 (iii), $\mathrm{lc}_\partial(F) = p\sigma^{i+1-k}(g)$ for some $p \in R[x]$. Then $\mathrm{lc}_\partial(aF) = \mathrm{lc}_\partial(p\partial^{i+1-k}T)$. It follows that $aF - p\partial^{i+1-k}T \in M_i$. Since

$$p\partial^{i+1-k}T \in R[x][\partial] \cdot M_k \subseteq R[x][\partial] \cdot M_i,$$

we have that $aF \in R[x][\partial] \cdot M_i$. On the other hand, $M_i \subset J$ by the induction hypothesis. Thus, $aF \in R[x][\partial] \cdot J$, which is J. Accordingly, $F \in J$ by the definition of saturation. □

4. AN ALGORITHM FOR COMPUTING CONTRACTION IDEALS

First, we translate an upper bound for the order of a desingularized operator over $Q_R[x]$ to $R[x]$.

Lemma 4.1. *Let $L \in R[x][\partial]$ with order $r > 0$, and $p \in R[x]$ be a primitive polynomial and a divisor of $\mathrm{lc}_\partial(L)$. Assume that there exists a p-removing operator for L over $Q_R[x]$. Then there exists a p-removing operator for L over $R[x]$ of the same order as the one over $Q_R[x]$.*

Proof. Assume that $P \in Q_R(x)[\partial]$ is a p-removing operator for L over $Q_R[x]$. Let P be of order k. Then PL is of the form

$$PL = \frac{a_{k+r}}{b_{k+r}}\partial^{k+r} + \cdots + \frac{a_1}{b_1}\partial + \frac{a_0}{b_0}$$

for some $a_i \in R[x], b_i \in R \setminus \{0\}$, $i = 0, \ldots, k+r$. Moreover,

$$\sigma^{-k}\left(\mathrm{lc}_\partial(PL)\right) = \frac{w}{vp}\,\mathrm{lc}_\partial(L),$$

where $w, v \in R[x]$ with $\gcd(w,p) = 1$.

Let $b = \mathrm{lcm}(b_0, b_1, \ldots, b_{k+r})$ in R and $P' = bP$. Then

$$P'L \in R[x][\partial] \quad \text{and} \quad \sigma^{-k}(\mathrm{lc}_\partial(PL)) = \frac{bw}{vp}\,\mathrm{lc}_\partial(L).$$

Since p is primitive, we have that $\gcd(bw, p) = 1$ in $R[x]$. Thus, P' is a p-removing operator of order k. □

By the above lemma, an order bound for a p-removing operator over $Q_R[x]$ is also an order bound for a p-removing operator over $R[x]$. The former has been well-studied in the literature. Order bounds for differential operators are given in [24, Algorithm 3.4] and [13, Lemma 4.3.12]. Those for recurrence operators are given in [7, Lemma 4] and [13, Lemma 4.3.3]. The quotients of desingularized operators with the given ideal generator are p-removing operators. So we can find order bounds for them.

By Theorem 3.3, determining a contraction ideal amounts to finding a desingularized operator T and a spanning set of M_k over $R[x]$, where k is an upper bound for the order of T.

Next, we present an algorithm for constructing a spanning set for $M_k(L)$ over $R[x]$, where L is a nonzero operator in $R[x][\partial]$ and k is a positive integer. To this end, we embed M_k into the free module $R[x]^{k+1}$ over $R[x]$.

Let us recall the right-hand division in $Q_R(x)[\partial]$ (see [5, Section 3] and [21, Page 483]). For each $F, G \in Q_R(x)[\partial]$ with $G \neq 0$, there exist unique elements $Q, R \in Q_R(x)[\partial]$ with $\deg_\partial(R) < \deg_\partial(G)$ or $R = 0$ such that $F = QG + R$. We call R the *right-hand remainder* of F by G and denote it by $\mathrm{rrem}(F, G)$.

Let $F \in R[x][\partial]$ with order k. Then $F \in M_k$ if and only if $F \in Q_R(x)[\partial]L$, which is equivalent to $\mathrm{rrem}(F, L) = 0$. Assume that $F = z_k\partial^k + \ldots + z_0$, where $z_k, \ldots, z_0 \in R[x]$ are to be determined. Then $\mathrm{rrem}(F, L) = 0$ gives rise to a linear system

$$(z_k, \ldots, z_0)A = \mathbf{0}, \tag{5}$$

where A is a $(k+1) \times r$ matrix over $Q_R(x)$. Clearing denominators of the elements in A, we may further assume that A is a matrix over $R[x]$. We are concerned with the solutions of (5) *over* $R[x]$. Set

$$N_k = \left\{(f_k, \ldots, f_0) \in R[x]^{k+1} \mid (f_k, \ldots, f_0)A = \mathbf{0}\right\}.$$

We call N_k the module of syzygies defined by (5). With the notation just specified, the following theorem is evident.

Theorem 4.2.

$$\begin{array}{cccc} \phi: & M_k & \longrightarrow & N_k \\ & \sum_{i=0}^k f_i\partial^i & \mapsto & (f_k, \ldots, f_0) \end{array}$$

is a module isomorphism over $R[x]$.

By Theorem 4.2, M_k is finitely generated over $R[x]$. To find a spanning set over of M_k over $R[x]$, it suffices to compute a spanning set of the module of syzygies defined by (5) over $R[x]$. When R is a field, we just need to solve (5) over a principal ideal domain [23, Chapter 5]. When R is the ring of integers or the ring of univariate polynomials over a field, we can use Gröbner bases of polynomials over a principal domain [14, 12]. Their implementations are available in computer algebra systems such as `Macaulay2` [12] and `Singular` [11].

We now consider how to construct a desingularized operator for L. For $k \in \mathbb{Z}^+$, we define

$$I_k = \left\{[\partial^k]P \mid P \in M_k\right\} \cup \{0\},$$

where $[\partial^k]P$ stands for the coefficient of ∂^k in P. It is clear that I_k is an ideal of $R[x]$. We call I_k the kth coefficient ideal of $\mathrm{Cont}(L)$. By the commutation rule, $\sigma(I_k) \subset I_{k+1}$.

Lemma 4.3. *Let $L \in R[x][\partial]$ be of positive order. If the kth submodule M_k of* $\mathrm{Cont}(L)$ *has a spanning set $\{B_1, \ldots, B_\ell\}$ over $R[x]$, then the kth coefficient ideal*

$$I_k = \left\langle [\partial^k]B_1, \ldots, [\partial^k]B_\ell \right\rangle.$$

Proof. Obviously, $\langle [\partial^k]B_1, \ldots, [\partial^k]B_\ell\rangle \subseteq I_k$. Let $f \in I_k$. Then $f = \mathrm{lc}_\partial(F)$ for some $F \in M_k$ with $\deg_\partial(F) = k$. Since M_k is generated by $\{B_1, \ldots, B_\ell\}$ over $R[x]$,

$$F = h_1B_1 + \cdots + h_\ell B_\ell, \quad \text{where } h_1, \ldots, h_\ell \in R[x].$$

Thus, $f = h_1\left([\partial^k]B_1\right) + \cdots + h_\ell\left([\partial^k]B_\ell\right)$. Consequently, $f \in \langle [\partial^k]B_1, \ldots, [\partial^k]B_\ell\rangle$. □

Theorem 4.4. *Let $L \in R[x][\partial]$ be of positive order. Assume that the kth submodule M_k of $\mathrm{Cont}(L)$ contains a desingularized operator for L. Let s be a nonzero element in the kth coefficient ideal with minimal degree. Then an operator S in M_k with leading coefficient s is a desingularized operator.*

Proof. Assume that T is a desingularized operator in M_k. By Lemma 3.2 (ii), we may assume that the order of T is equal to k. Let $t = \mathrm{lc}_\partial(T)$. Then $\deg(t) = \deg(s)$ by Lemma 3.2 (i). Let u be the leading coefficient of s with respect to x and v be that of t. Then $ut - vs$ is zero. Otherwise, $uT - vS$ would be an operator of order k whose leading coefficient with respect to ∂ has degree lower than $\deg_x(t)$, a contradiction to Lemma 3.2 (i). It follows from $ut = vs$ and Definition 3.1 that S is a desingularized operator. □

Let L be an operator in $R[x][\partial]$ of positive order. We can compute a spanning set $\{B_1, \ldots, B_\ell\}$ for the kth submodule of $\mathrm{Cont}(L)$ by Theorem 4.2, where k is an upper bound on the order of a desingularized operator for L.

Set $b_i = [\partial^k]B_i$, $i = 1, \ldots, \ell$. By Lemma 4.3, the kth coefficient ideal I_k of $\mathrm{Cont}(L)$ is generated by $\{b_1, \ldots, b_\ell\}$. Let $\bar{I}_k$ be the extension ideal of I_k in $Q_R[x]$. Since $Q_R[x]$ is a principal ideal domain, we have that $\bar{I}_k = \langle s'\rangle$ for some $s' \in Q_R[x]$. Then there exist $c'_1, \ldots, c'_\ell \in Q_R[x]$ such that $c'_1 b_1 + \ldots + c'_\ell b_\ell = s'$. By clearing denominators, we can find $c_1, \ldots, c_\ell \in R[x]$ such that $c_1 b_1 + \cdots + c_\ell b_\ell = s$, where $s = cs'$ for some $c \in R$. Then s is an element in I_k with minimal degree. It follows from Theorem 4.4 that $T = c_1 B_1 + \cdots + c_\ell B_\ell$ is a desingularized operator for L with $\mathrm{lc}_\partial(T) = s$.

Let c be the content of s. By Theorem 3.3, $\mathrm{Cont}(L)$ is the saturation of $R[x][\partial] \cdot M_k$ with respect to c. Note that c is contained in the center of $R[x][\partial]$. So a basis of the saturation ideal can be computed in the same way as in the commutative case.

To this end, we need to introduce some new indeterminates. Let σ_y be the identity map of $R[x, y]$, where y is a new indeterminate. Let δ_y be the σ_y-derivation that maps everything in $R[x, y]$ to zero. Then one can extend the ring $R[x][\partial]$ to $R[x, y][\partial, \partial_y]$. Moreover, $R[y][\partial_y]$ lies in the center of the extended ring. For $r \in R$, one can define an evaluation map

$$\begin{array}{cccc} \phi_r: & R[x,y][\partial,\partial_y] & \longrightarrow & R[x][\partial] \\ & \sum_{i=0}^{\ell}\sum_{j=0}^{m} f_{ij} y^i \partial_y^j & \mapsto & \sum_{i=0}^{\ell} f_{i0} r^i, \end{array}$$

where $f_{ij} \in R[x][\partial]$. Since $R[y][\partial_y]$ is contained in the center of $R[x, y][\partial, \partial_y]$, the map ϕ_r is a ring homomorphism.

Proposition 4.5. *Let I be a left ideal of $R[x][\partial]$ and c be non-zero element in R. Assume that J is a left ideal*

$$R[x,y][\partial,\partial_y] \cdot (I \cup \{1 - cy\}),$$

Then $I : c^\infty = J \cap R[x][\partial]$.

Proof. Let $J_{x,\partial} = J \cap R[x][\partial]$. If $G \in J_{x,\partial}$, then

$$G = Q_1 P + Q_2(1 - cy) \tag{6}$$

with $Q_1, Q_2 \in R[x, y][\partial, \partial_y]$ and $P \in I$. Temporarily passing to the extended ring $Q_R[x, y][\partial, \partial_y]$ of $R[x, y][\partial]$, we may apply the evaluation homomorphism $\phi_{1/c}$ to (6) and then multiply the equation by c^d, where $d = \deg_y(Q_1)$. We thus obtain $c^d G = QP$ with Q being in $R[x][\partial]$. Consequently, $J_{x,\partial} \subset I : c^\infty$.

Conversely, let $G \in I : c^\infty$, say $c^d G \in I$. Then $G \in R[x][\partial]$ and $c^d G \in J$. Since $1 - cy$ belongs to J,

$$1 - (cy)^d = (1 + cy + (cy)^2 + \cdots + (cy)^{d-1})(1 - cy) \in J$$

Since y and c commute with every element of $R[x, y][\partial]$,

$$\left(1 - (cy)^d\right) G = G\left(1 - (cy)^d\right) \in J.$$

Again, $(cy)^d G = y^d(c^d G) \in J$ because $c^d G \in J$. It follows that $G = \left(1 - (cy)^d\right) G + (cy)^d G \in J$. Thus, $G \in J_{x,\partial}$. □

For the case $R = \mathbb{Q}[t]$, where t is an indeterminate, an implementation of a saturation ideal with respect to a constant is available in [16].

We outline our method for determining contraction ideals.

Algorithm 4.6. *Given $L \in R[x][\partial]$, where $\partial x = (x + 1)\partial$ or $\partial x = x\partial + 1$, compute a basis of $\mathrm{Cont}(L)$.*

(1) Derive an upper bound k on the order of a desingularized operator for L.

(2) Compute a spanning set of M_k over $R[x]$.

(3) Compute a desingularized operator T, and set a to be the content of $\mathrm{lc}_\partial(T)$.

(4) Compute a basis of $(R[x][\partial] \cdot M_k) : a^\infty$.

The termination of this algorithm is evident. Its correctness follows from Theorem 3.3. We assume that the commutation rule in $R[x][\partial]$ is either $\partial x = (x+1)\partial$ or $\partial x = x\partial + 1$ in $R[x][\partial]$, because we only know order bounds for those cases. In the shift case, a bound is derived from [7, Lemma 4]. More concretely, we factor $\mathrm{lc}_\partial(L)$ and compute the maximum of the dispersions of the factors with the trailing coefficient. In the differential case, we need to follow steps 1, 2 and 3 in [24, Algorithm 3.4] to rewrite the input operator as a b-function in the algebraic extension of each factor of $\mathrm{lc}_\partial(L)$, and bound integer roots of the trailing coefficient in each b-function. In step 2, we need to solve linear systems over $R[x]$ as stated in Theorem 4.2. This can be done by Gröbner basis computation. In step 3, T is computed according to Theorem 4.4 and the extended Euclidean algorithm in $Q_R[x]$. The last step is carried out according to Proposition 4.5.

Example 4.7. *Let $\mathbb{Q}[t][n][\partial]$ be the shift Ore algebra, in which the commutation rule is $\partial n = (n + 1)\partial$. Consider*

$$L = (n - 1)(n + t)\partial + n + t + 1.$$

By [7, Lemma 4], we obtain an order bound 2 for a desingularized operator. Thus, M_2 contains a desingularized operator for L. In step 2 of Algorithm 4.6, we find that M_2 is generated by

$$\begin{aligned} T_1 &= (2 + t)n\partial^2 + (4 - n + t)\partial - 1, \\ T_2 &= (n - 1)n\partial^2 + 2(n - 1)\partial + 1, \end{aligned}$$

where T_1 is a desingularized operator, $\mathrm{lc}_\partial(T_1) = (2 + t)n$. Using Gröbner bases, $\mathrm{Cont}(L) = (\mathbb{Q}[t][n][\partial] \cdot M_2) : (2 + t)^\infty$ is generated by $\{L, T_1\}$.

Let us consider the example in Section 1.

Example 4.8. *In the shift Ore algebra* $\mathbb{Z}[n][\partial]$, *let*

$$L = (1+16n)^2\partial^2 - 32(7+16n)\partial - (1+n)(17+16n)^2.$$

By [7, Lemma 4], we obtain an order bound 3 *for a desingularized operator. Thus,* M_3 *contains a desingularized operator for* L. *In step 2 of Algorithm 4.6, we find that* M_3 *is generated by* $\{L, \tilde{T}\}$, *where* $\tilde{T}$ *is given in* (1). *Note that* $\mathrm{lc}_\partial(\tilde{T})=1$. *Thus,* $\tilde{T}$ *is a desingularized operator. Consequently,*

$$\mathrm{Cont}(L) = (\mathbb{Z}[n][\partial]\cdot\{L,\tilde{T}\}) : 1^\infty = \mathbb{Z}[n][\partial]\cdot\{L,\tilde{T}\}.$$

Example 4.9. *Let* $\mathbb{Z}[x][\partial]$ *be the differential Ore algebra, in which the commutation rule is* $\partial x = x\partial + 1$. *Consider the operator* $L = x\partial^2 - (x+2)\partial + 2 \in \mathbb{Z}[x][\partial]$ *in [3]. By [24, Algorithm 3.4], we obtain an order bound* 4 *for a desingularized operator. Thus,* M_4 *contains a desingularized operator for* L. *In step 2 of Algorithm 4.6, we find that* M_4 *is generated by* $\{L, \partial L, T\}$, *where* $T = \partial^4 - \partial^3$. *Note that* $\mathrm{lc}_\partial(T) = 1$. *Thus,* T *is a desingularized operator. Consequently,*

$$\mathrm{Cont}(L) = (\mathbb{Z}[x][\partial]\cdot\{L,\partial L,T\}) : 1^\infty = \mathbb{Z}[x][\partial]\cdot\{L,T\}.$$

5. COMPLETE DESINGULARIZATION

As seen in Section 1, the shift operator

$$L = (1+16n)^2\partial^2 - (224+512n)\partial - (1+n)(17+16n)^2$$

has a desingularized operator T with leading coefficient 64. But the content of $\mathrm{lc}_\partial(L)$ is 1. The redundant content 64 has been removed by computing another desingularized operator $\tilde{T}$ in (1). This enables us to see immediately that the sequence annihilated by L is an integer sequence when its initial values are integers.

Krattenthaler proposes a conjecture in [17]: Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two P-recursive sequences over $\mathbb{Z}$ with leading coefficients n. Then $(n!a_nb_n)_{n\geq 0}$ is also a P-recursive sequence over $\mathbb{Z}$ with leading coefficient n. To test the conjecture for the two particular sequences, one may first compute an annihilator L of $(n!a_nb_n)_{n\geq 0}$, and then look for a nonzero operator in $\mathrm{Cont}(L)$ whose leading coefficient has both minimal degree and "minimal" content with respect to n. When the content is equal to 1, the conjecture is true for these sequences.

These two observations motivate us to define the notion of completely desingularized operators.

Definition 5.1. *Let* $L \in R[x][\partial]$ *with positive order, and* Q *a desingularized operator for* L. *Set* c *be the content of* $\mathrm{lc}_\partial(Q)$. *We call* Q *a* completely desingularized operator *for* L *if* c *is a divisor of the content of the leading coefficient of every desingularized operator for* L.

To see the existence of completely desingularized operators, we assume that L is of order r. For a desingularized operator T of order k, equations (3) and (4) in Definition 3.1 enable us to write

$$\sigma^{r-k}(\mathrm{lc}_\partial(T)) = c_T\, g, \tag{7}$$

where $c_T \in R$ and $g = p_1^{e_1-k_1}\cdots p_s^{e_m-k_m}$. Note that g is primitive and independent of the choice of desingularized operators.

Lemma 5.2. *Let* $L \in R[x][\partial]$ *with order* $r > 0$. *Set* I *to be the set consisting of zero and* c_T *given in* (7) *for all desingularized operators for* L. *Then* I *is an ideal of* R.

Proof. By Definition 3.1, the product of a nonzero element of R and a desingularized operator for L is also a desingularized one. So it suffices to show that I is closed under addition. Let T_1 and T_2 be two desingularized operators of orders k_1 and k_2, respectively. Assume that $k_1 \geq k_2$. By (7),

$$\sigma^{r-k_1}(\mathrm{lc}_\partial(T_1)) = c_1\, g \quad\text{and}\quad \sigma^{r-k_2}(\mathrm{lc}_\partial(T_2)) = c_2\, g.$$

If $c_1 + c_2 = 0$, then there is nothing to prove. Otherwise, a direct calculation implies that

$$\mathrm{lc}_\partial(T_1) = c_1\sigma^{k_1-r}(g) \quad\text{and}\quad \mathrm{lc}_\partial\left(\partial^{k_1-k_2}T_2\right) = c_2\sigma^{k_1-r}(g).$$

Thus, $T_1+\partial^{k_1-k_2}T_2$ has leading coefficient $(c_1+c_2)\sigma^{k_1-r}(g)$. Accordingly, $T_1 + \partial^{k_1-k_2}T_2$ is a desingularized one, which implies that $c_1 + c_2$ belongs to I. □

Since R is a principal ideal domain, I in the above lemma is generated by an element c, which corresponds to a completely desingularized operator.

The next technical lemma serves as a step-stone to construct completely desingularized operators.

Lemma 5.3. *Let* $L \in R[x][\partial]$ *with order* $r > 0$, *and* $k \geq r$. *Then* $R[x][\partial]\cdot M_k = R[x][\partial]\cdot M_{k+1}$ *if and only if* $\sigma(I_k) = I_{k+1}$.

Proof. Assume that $\sigma(I_k) = I_{k+1}$. Since $M_k \subset M_{k+1}$, it suffices to prove that $M_{k+1} \subset R[x][\partial]\cdot M_k$.

For each $T \in M_{k+1}\setminus M_k$, we have that $\mathrm{lc}_\partial(T) \in \sigma(I_k)$. Thus, there exists $F \in M_k$ such that $\sigma(\mathrm{lc}_\partial(F)) = \mathrm{lc}_\partial(T)$. In other words, $T - \partial F \in M_k$. Consequently, $T \in R[x][\partial]\cdot M_k$.

Conversely, assume that $R[x][\partial]\cdot M_{k+1} = R[x][\partial]\cdot M_k$. It suffices to prove that $I_{k+1} \subseteq \sigma(I_k)$ because $\sigma(I_k) \subseteq I_{k+1}$ by definition. Let $\mathcal{B}$ be a spanning set of M_k over R[x]. Then $\mathcal{B}$ is also a basis of the left ideal $R[x][\partial]\cdot M_k$.

Let $\prec$ be the term order such that $x^{\ell_1}\partial^{m_1} \prec x^{\ell_2}\partial^{m_2}$ if either $m_1 < m_2$ or $m_1 = m_2$ and $\ell_1 < \ell_2$. Since $\deg_\partial(P) \leq k$ for each $P \in \mathcal{B}$, S-polynomials and G-polynomials formed by elements in M_k have orders no more than k [4, Definition 10.9]. By Buchberger's algorithm, there exists a Gröbner basis $\mathcal{G}$ of $R[x][\partial]\cdot\mathcal{B}$ with respect to $\prec$ such that $\deg_\partial(G) \leq k$ for each $G \in \mathcal{G}$.

For $p \in I_{k+1}\setminus\{0\}$, there exists $T \in M_{k+1}\setminus M_k$ such that $\mathrm{lc}_\partial(T) = p$. Since $T \in R[x][\partial]\cdot M_{k+1}$, we have $T \in R[x][\partial]\cdot M_k$. It follows that T is reduced to zero by $\mathcal{G}$. Thus,

$$T = \sum_{G\in\mathcal{G}} V_G G \quad \text{with } \mathrm{HT}(V_G G) \preceq \mathrm{HT}(T). \tag{8}$$

By the choice of term order, $\deg_\partial(V_G G) \leq k+1$. If $V_G G$ is of order $k+1$, then $\mathrm{lc}_\partial(V_G G) = a_G\,\sigma^{k+1-d_G}(\mathrm{lc}_\partial(G))$, where a_G is in $R[x]$ and d_G is the order of G. Comparing the leading coefficients of operators in both sides of (8) and noticing $\deg_\partial(T) = k+1$, we have

$$p = \sum_{\deg_\partial(V_G G) = k+1} a_G\,\sigma^{k+1-d_G}(\mathrm{lc}_\partial(G)).$$

It follows that

$$\sigma^{-1}(p) = \sum_{\deg_\partial(V_G G) = k+1} \sigma^{-1}(a_G)\,\sigma^{k-d_G}(\mathrm{lc}_\partial(G)). \tag{9}$$

On the other hand, $\sigma^{k-d_G}(\mathrm{lc}_\partial(G)) = \mathrm{lc}_\partial\left(\partial^{k-d_G}G\right)$ implies that $\sigma^{k-d_G}(\mathrm{lc}_\partial(G)) \in I_k$. We have that $\sigma^{-1}(p) \in I_k$ by (9). Thus, $I_{k+1} \subset \sigma(I_k)$. □

By the above lemma, $I_j = \sigma^{j-\ell}(I_\ell)$ whenever $j \geq \ell$ and $\text{Cont}(L) = R[x][\partial] \cdot M_\ell$. In this case, a basis of I_j can be obtain by shifting a basis of I_ℓ, which allows us to find a completely desingularized operator.

Theorem 5.4. *Let $L \in R[x][\partial]$ with order $r > 0$. Assume that the ℓth submodule M_ℓ of $\text{Cont}(L)$ contains a basis of $\text{Cont}(L)$. Let I_ℓ be the ℓth coefficient ideal of $\text{Cont}(L)$, and $\mathbf{G}$ a reduced Gröbner basis of I_ℓ. Let $f \in \mathbf{G}$ be of the lowest degree in x and F be the operator in $\text{Cont}(L)$ with $\text{lc}_\partial(F) = f$. Then F is a completely desingularized operator for L.*

Proof. By Lemma 5.2, $\text{Cont}(L)$ contains a completely desingularized operator S. Let $j = \deg_\partial(S)$. Then $\text{lc}_\partial(S)$ is in I_j for some $j \geq \ell$. By Lemma 5.3, $\sigma^{j-\ell}(I_\ell) = I_j$. It follows that $\sigma^{\ell-j}(\text{lc}_\partial(S))$ belongs to I_ℓ. By (7), we have

$$\sigma^{r-j}\left(\text{lc}_\partial(S)\right) = c_S\, g,$$

where $c_S \in R$ and g is a primitive polynomial in $R[x]$. A direct calculation implies that $\sigma^{\ell-j}(\text{lc}_\partial(S)) = c_S\sigma^{\ell-r}(g)$. Since $\sigma^{\ell-j}(\text{lc}_\partial(S)) \in I_\ell$, so does $c_S\sigma^{\ell-r}(g)$.

Note that F is a desingularized operator by Theorem 4.4. By (7), $\sigma^{r-\ell}(f) = c_F\, g$, where $c_F \in R$. Thus, $f = c_F\sigma^{\ell-r}(g)$.

Since $\mathbf{G}$ is a reduced Gröbner basis of I_ℓ, f is the unique polynomial in $\mathbf{G}$ with minimal degree. Moreover, $c_S\sigma^{l-r}(g)$ is of minimal degree in I_ℓ. So it can be reduced to zero by f. Thus, $c_F \mid c_S$. On the other hand, $c_S \mid c_F$ by Definition 5.1. Thus, c_S and c_F are associated to each other. Consequently, F is a completely desingularized operator for L. □

The construction in the above theorem leads to the following algorithm.

Algorithm 5.5. *Given $L \in R[x][\partial]$, where $\partial x = (x+1)\partial$ or $\partial x = x\partial + 1$, compute a completely desingularized operator for L.*

(1) Compute a basis $\mathcal{A}$ of $\text{Cont}(L)$ by Algorithm 4.6.

(2) Set ℓ to be the highest order of the elements in $\mathcal{A}$. Compute a spanning set of M_ℓ over $R[x]$.

(3) Set $\mathcal{B}' = \{B \in \mathcal{B} \mid \deg_\partial(B) = \ell\}$. Compute a reduced Gröbner basis $\mathbf{G}$ of $\langle \{\text{lc}_\partial(B) \mid B \in \mathcal{B}'\}\rangle$.

(4) Set f to be the polynomial in $\mathbf{G}$ whose degree is the lowest one in x. Tracing back to the computation of step 3, one can find $u_B \in R[x]$ such that $f = \sum_{B\in\mathcal{B}'} u_B\,\text{lc}_\partial(B)$.

(5) Output $\sum_{B\in\mathcal{B}'} u_B B$.

The termination of this algorithm is evident. Its correctness follows from Theorem 5.4.

Example 5.6. *Consider two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ satisfying the following two recurrence equations [17]*

$$na_n = a_{n-1} + a_{n-2} \quad \text{and} \quad nb_n = b_{n-1} + b_{n-5},$$

respectively. The sequence $c_n = n!a_nb_n$ has an annihilator $L \in \mathbb{Z}[n][\partial]$ with

$$\deg_\partial(L) = 10 \text{ and } \text{lc}_\partial(L) = (n+10)(n^6+47n^5+\cdots+211696).$$

In step 1 of the above algorithm, $\text{Cont}(L) = R[x][\partial] \cdot M_{14}$. In steps 2 and 3, we observe that I_{14} is generated by $n+14$. In other words, we obtain a completely desingularized operator T of order 14 with $\text{lc}_\partial(T) = n + 14$. Translating into the recurrence equations of c_n, we arrive at

$$nc_n = \alpha_1 c_{n-1} + \cdots + \alpha_{14}c_{n-14},$$

where $\alpha_i \in \mathbb{Z}[n]$, $i = 1, \ldots, 14$. This verifies Krattenthaler's conjecture for the sequences a_n and b_n.

Note that it is impossible to have a completely desingularized operator of order less than 14. In fact, for some lower orders, one can obtain

$$\begin{aligned}
\sigma^{-11}(I_{11}) &= \langle 11104n, 4n(n-466), n(n^2-34n+1336)\rangle,\\
\sigma^{-12}(I_{12}) &= \langle 4n, n(n-24)\rangle,\\
\sigma^{-13}(I_{13}) &= \langle 2n, n(n-26)\rangle.
\end{aligned}$$

They cannot produce a leading coefficient whose degree and content are both minimal.

Example 5.7. *Consider the following recurrence equations:*

$$\begin{aligned}
na_n &= (31n-6)a_{n-1} + (49n-110)a_{n-2} + (9n-225)a_{n-3},\\
nb_n &= (4n+13)b_{n-1} + (69n-122)b_{n-2} + (36n-67)b_{n-3}.
\end{aligned}$$

Let $c_n = n!a_nb_n$, which has an annihilator $L \in \mathbb{Z}[n][\partial]$ of order 10 with $\text{lc}_\partial(L) = (n+9)\alpha$, where $\alpha \in \mathbb{Z}[n]$ and $\deg_n(\alpha) = 20$.

By the known algorithms for desingularization in [2, 1, 7, 8], we find that c_n satisfies the recurrence equation

$$\beta nc_n = \beta_1 c_{n-1} + \ldots + \beta_{10}c_{n-10},$$

where β is an 853-digit integer, $\beta_i \in \mathbb{Z}[n]$, $i = 1, \ldots, 10$.

On the other hand, Algorithm 5.5 finds a completely desingularized operator T for L of order 14 whose leading coefficient is $n + 14$. Translating into the recurrence equation of c_n yields $nc_n = \gamma_1 c_{n-1} + \cdots + \gamma_{14}c_{n-14}$, where $\gamma_i \in \mathbb{Z}[n]$.

Let $L \in R[x][\partial]$ with positive order and T a desingularized operator for L. Then the degree of $\text{lc}_\partial(T)$ in x is equal to

$$d = \deg_x\left(\text{lc}_\partial(L)\right) - \left(\deg_x(p_1)k_1 + \cdots + \deg_x(p_m)k_m\right),$$

where $k_1, \ldots, k_m$ are given in Definition 3.1. Hence, $\text{Cont}(L)$ cannot contain any operator whose leading coefficient has degree lower than d.

We provide a lower bound for the content of the leading coefficients of operators in $\text{Cont}(L)$ with respect to the divisibility relation on R. To this end, we write

$$L = a_k f_k(x)\partial^k + a_{k-1}f_{k-1}(x)\partial^{k-1} + \cdots + a_0 f_0(x)$$

where $a_i \in R$ and $f_i(x) \in R[x]$ is primitive, $i = 0, 1, \ldots, k$. We say that L is *R-primitive* if $\gcd(a_0, a_1, \ldots, a_k) = 1$. As an easy consequence of [9, Lemma 9.5], Gauss's lemma in the commutative case also holds for R-primitive polynomials.

Lemma 5.8. *Let P and Q be two operators in $R[x][\partial]$. If P and Q are R-primitive, so is PQ.*

Theorem 5.9. *Let $L \in R[x][\partial]$ with positive order and p be a non-unit element of R. If L is R-primitive and $p \mid \text{lc}_\partial(L)$, then p is non-removable.*

Proof. Assume that p is removable, then there exists a p-removing operator P such that $PL \in R[x][\partial]$. By Lemma 2.4, we can write

$$P = \frac{p_0}{p^{d_0}} + \frac{p_1}{p^{d_1}}\partial + \cdots + \frac{p_k}{p^{d_k}}\partial^k$$

where $p_i \in R[x]$, $\gcd(p_i, p) = 1$ in $R[x]$, $i = 0, \ldots, k$ and $d_k \geq 1$. Let $d = \max_{0\leq i\leq k} d_i$ and $P_1 = p^d P$. Then the content c of P_1 with respect to ∂ is $\gcd(p_0, \ldots, p_k)$ because $\gcd(p_i, p) = 1$, $i = 0, \ldots, k$. Let $P_1 = cP_2$. Then P_2 is the primitive part of P_1. In particular, P_2 is R-primitive. Then $cP_2L = p^dPL$. Since $\gcd(c, p) = 1$ and $PL \in R[x][\partial]$, p divides the content of P_2L with respect to ∂. Since p is a non-unit element of R, P_2L is not R-primitive, a contradiction to Lemma 5.8. □

Example 5.10. *The minimal annihilator of* $\binom{4n}{n} + 3^n$ *is of order 2. Its leading coefficient is*

$$3(n+2)(3n+4)(3n+5)(7n+3)\left(25n^2+21n+2\right)$$

We observe that 3 *is a constant factor of the leading coefficient. By Theorem 5.9,* 3 *is non-removable.*

6. CONCLUDING REMARKS

In this paper, we determine a basis of a contraction ideal defined by an Ore operator in $R[x][\partial]$, and compute a completely desingularized operator whose leading coefficient is minimal in terms of both degree and content. A more challenging topic is to consider the corresponding problems in the multivariate case.

Our algorithms rely heavily on the computation of Gröbner bases over a principal ideal domain R. At present, the computation of Gröbner bases over R is not fully available in a computer algebra system. So the algorithms in this paper are not yet implemented. To improve their efficiency, we need to use linear algebra over R as much as possible.

7. ACKNOWLEDGEMENT

I am grateful to my supervisors Manuel Kauers and Ziming Li for initiating this research project, stimulating inspirational discussions and helping me revise the paper. I thank Shaoshi Chen and anonymous referees for valuable information on literatures.

8. REFERENCES

[1] S. A. Abramov, M. Barkatou, and M. van Hoeij. Apparent singularities of linear difference equations with polynomial coefficients. *AAECC*, 117–133, 2006.

[2] S. A. Abramov and M. van Hoeij. Desingularization of linear difference operators with polynomial coefficients. In *Proc. of ISSAC'99*, 269–275, New York, NY, USA, 1999, ACM.

[3] M. A. Barkatou and S. S. Maddah. Removing apparent singularities of systems of linear differential equations with rational function coefficients. In *Proc. of ISSAC'15*, 53–60, New York, NY, USA, 2015, ACM.

[4] T. Becker and V. Weispfenning. *Gröbner Bases, A Computational Approach to Commutative Algebra.* Springer-Verlag, New York, USA, 1993.

[5] M. Bronstein and M. Petkovšek. An introduction to pseudo-linear algebra. *Theoretical Computer Science*, 3 –33, 1996.

[6] M. Ceria and T. Mora. Buchberger-Zacharias theory of multivariate Ore extensions. Submitted to J. of Pure and Applied Algebra., 2015.

[7] S. Chen, M. Jaroschek, M. Kauers, and M. F. Singer. Desingularization explains order-degree curves for Ore operators. In *Proc. of ISSAC'13*, 157–164, New York, NY, USA, 2013, ACM.

[8] S. Chen, M. Kauers, and M. F. Singer. Desingularization of Ore operators. *J. Symb. Comput.*, 74:617–626, 2016.

[9] R. C. Churchill and Y. Zhang. Irreducibility criteria for skew polynomials. *Journal of Algebra*, 322:3797–3822, 2009.

[10] F. Chyzak and B. Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. *J. Symb. Comput.* , 26:187–227, 1998.

[11] W. Decker et al. *Singular, a computer algebra system for polynomial computations.* `https://www.singular.uni-kl.de/`.

[12] D. Grayson, M. Stillman, and D. Eisenbud. Macaulay2, a software system for research in algebraic geometry. `http://www.math.uiuc.edu/Macaulay2/`.

[13] M. Jaroschek. *Removable Singularities of Ore Operators.* PhD thesis, RISC-Linz, Johannes Kepler Univ., 2013.

[14] A. Kandri-Rody and D. Kapur. Computing a Gröbner basis of a polynomial ideal over a Euclidean domain. *J. Symb. Comput.* , 6:37–57, 1988.

[15] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. *J. Symb. Comput.* , 9:1–26, 1990.

[16] C. Koutschan. *HolonomicFunctions (User's Guide).* RISC Report Series, Johannes Kepler Univ., 2010.

[17] G. Labahn et al. Workshop on symbolic combinatorics and computational differential algebra. `http://www.fields.utoronto.ca/video-archive/event/411/2015`.

[18] Z. Li and Y. Zhang. A note on Groebner bases of Ore polynomials over a PID. `http://www.algebra.uni-linz.ac.at/people/yzhang/GB.pdf`.

[19] J. Middeke. *A computational view on normal forms of matrices of Ore polynomials.* PhD thesis, Johannes Kepler University Linz, 2011.

[20] T. Mora. *Solving Polynomial Equation Systems* Vol. IV. Cambridge University Press, 2016.

[21] Ø. Ore. Theory of non-commutative polynomials. *The Annal of Mathematics*, 34:480–508, 1933.

[22] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations.* Springer-Verlag, New York, USA, 1999.

[23] A. Storjohann. *Algorithms for Matrix Canonical Forms.* PhD thesis, Swiss Federal Institute of Technology Zurich, 2000.

[24] H. Tsai. Weyl closure of a linear differential operator. *J. Symb. Comput.* , 29:747–775, 2000.