Formal Power Series Solutions of Algebraic Ordinary Differential Equations

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Abstract

In this paper, we consider nonlinear algebraic ordinary differential equations (AODEs) and study their formal power series solutions. Our method is inherited from Lemma 2.2 in [J. Denef and L. Lipshitz, *Power series solutions of algebraic differential equations*, Mathematische Annalen, **267**(1984), 213-238] for expressing high order derivatives of a differential polynomial via their lower order ones. By a careful computation, we give an explicit formula for the expression. As an application, we give a method for determining the existence of a formal power series solution with given first coefficients. We define a class of certain differential polynomials in which our method works properly, which is called *non-vanishing*. A statistical investigation shows that many differential polynomials in the literature are non-vanishing.

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. An algebraic ordinary differential equation (AODE) is an equation of the form

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0,$$

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for some $n \in \mathbb{N}$ and F a polynomial in $n+2$ variables over $\mathbb{K}$. This paper addresses a study of formal power series solutions, i.e. solutions in $\mathbb{K}[[x]]$, of nonlinear AODEs.

The problem of finding formal power series solutions of AODEs has a long history and it has been heavily studied in the literature. The Newton polygon method is a well-known method developed for studying this problem. In [BB56], Briot and Bouquet use the Newton polygon method for studying the singularities of first-order and first degree ODEs. Fine gave a generalization of the method for arbitrary order AODEs in [Fin89]. By using Newton polygon method, one can obtain interesting results on a larger class of series solutions which is called generalized formal power series solutions, i.e. power series with real exponents. In [GS91], Grigoriev and Singer proposed a parametric version of the Newton polygon method and use it to study generalized formal power series solutions of AODEs. A worth interpretation of the parametric Newton polygon method can be found in [CF09] and [Can05]. However, it has been shown in [DRJ97] that Newton polygon method for AODEs has its own limits in the sense that in some cases, it fails to give a solution.

In this paper, we follow a different method which is inherited from the work by Denef and Lipshitz in [DL84]. In [DL84] the authors presented a possibly algorithm for determining the existence of a formal power series solution of a system of AODEs. One of the fundamental steps in their construction is the expression of $F^{(2k+2+q)}$ in terms of lower order AODEs for arbitrary natural numbers k, q (see [DL84, Lemma 2.2]). In this paper, by a careful computation, we present an explicit formula for the expression (Theorem 3.6).

As a nice application, we use the explicit formula to decide the existence of formal power series solutions of a given AODE whose first coefficients are given, and in the affirmative case, compute all of them (Theorem 4.5 and 5.6). If the AODE is given together with a suitable initial value data, or equivalently suitable first coefficients, then one can decide immediately the existence of a formal power series solution and determine such a solution by Cauchy method (Proposition 2.3). However, there are formal power series solutions which cannot be determined by Cauchy method. The method we developed in this paper can be used to overcome this difficulty. Moreover, we can determine all (truncated) formal power series solutions of a certain class of AODEs.

Another interesting application of the method is to give an explicit statement of a result by Hurwitz in [Hur89]. Hurwitz proved that if z is a formal power series solution of an AODE, then there exists a large enough positive integer N such that coefficients of order $\geq N$ are determined by a recursion formula. Our result (Algorithm 4.6 and 5.7) can be used to determine a sharp upper bound for such an N and the recursion formula (compare with [Hoe14]).

The rest of the paper is organized as follow. Section 2 is to give necessary notations and definitions for the rest of the paper. We give a refinement for a lemma by Denef and Lipshitz in [DL84] in Section 3. Section 4 and 5 present an application of Theorem 3.6 to the problem of deciding the existence of a formal power series with given first coefficients. In Section 5, we define certain AODEs in which our method can work properly. They are called *non-vanishing* AODEs. Section 6 is devoted for statistical investigation of non-vanishing AODEs in the literature.

2 Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $n \in \mathbb{N}$. Assume that $\mathbb{K}[x]\{y\}$ ($\mathbb{K}(x)\{y\}$) is the ring of differential polynomials in y with coefficients in the ring of polynomials $\mathbb{K}[x]$ (the field of rational functions $\mathbb{K}(x)$, respectively), where the derivation of x is 1. A differential polynomial is of order $n \geq 0$ if the n-th derivative $y^{(n)}$ is the highest derivative appearing in it.

Consider the algebraic ordinary differential equation (AODE) of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where F is a differential polynomial in $\mathbb{K}(x)\{y\}$ of order n. For simplicity, we may also write (1) as $F(y) = 0$, and call n the *order* of (1). As a matter of notation, we set

$$\frac{\partial F}{\partial y^{(k)}} = 0 \;\text{ if }\; k < 0. \tag{2}$$

Next, we recall a lemma concerning the k-th derivative of F with respect to x (see [Rit50, page 30]).

Lemma 2.1. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \geq 0$. Then for each $k \geq 1$, there exists a differential polynomial $R_k \in \mathbb{K}[x]\{y\}$ of order at most $n + k - 1$ such that*

$$F^{(k)} = S_F \cdot y^{(n+k)} + R_k, \tag{3}$$

where $S_F = \frac{\partial F}{\partial y^{(n)}}$ is the separant of F.

Let $\mathbb{K}[[x]]$ be the ring of formal power series with respect to x. For each formal power series $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$, we use the notation $[x^k]f$ to refer the coefficient of x^k in f. The coefficient of x^k in a formal power series can be translated into the constant coefficient of its k-th formal derivative, as stated in the following lemma (see [KP10, Theorem 2.3, page 20]).

Lemma 2.2. *Let $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$. Then $[x^k]f = [x^0]\left(\frac{1}{k!} f^{(k)}\right)$.*

Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n and $\mathbf{c} = (c_0, c_1, \ldots)$ be a tuple of indeterminates or elements in $\mathbb{K}$. As a notation we set $F(\mathbf{c}) = F(0, c_0, ..., c_n)$.

Assume that $z = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is a formal power series solution of the AODE $F(y) = 0$ at the origin, where $c_i \in \mathbb{K}$ is unknown. Set $\mathbf{c} = (c_0, c_1, \ldots)$. By Lemma 2.2, we know that $F(z) = 0$ if and only if $[x^0]F^{(k)}(z) = F^{(k)}(\mathbf{c}) = 0$ for each $k \geq 0$.

Based on the above fact and Lemma 2.1, we have the following proposition.

Proposition 2.3. [1] *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\tilde{\mathbf{c}} = (c_0, c_1, \ldots, c_n) \in \mathbb{K}^{n+1}$ satisfies $F(\tilde{\mathbf{c}}) = 0$ and $S_F(\tilde{\mathbf{c}}) \neq 0$. For $k > 0$, set*

$$c_{n+k} = -\frac{R_k(0, c_0, \ldots, c_{n+k-1})}{S_F(\tilde{\mathbf{c}})},$$

where R_k is specified in Lemma 2.1. Then $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a solution of $F(y) = 0$.

In the above proposition, if the initial value $\tilde{\mathbf{c}}$ vanishes at the separant of F, we may expand R_k in Lemma 2.1 further to get formal power series solutions, as the following example illustrates.

Example 2.4. Consider the following AODE:

$$F = xy' + y^2 - y - x^2 = 0.$$

Since $S_F = x$, we cannot apply Proposition 2.3 to get a formal power series solutions of the above AODE. Instead, we observe from computation that

$$F^{(k)} = xy^{(k+1)} + (2y + k - 1)y^{(k)} + \tilde{R}_{k-1}, \tag{4}$$

where $\tilde{R}_{k-1} \in \mathbb{K}[x]\{y\}$ is of order $k - 1$ and $k \geq 1$.

Assume that $z = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is a formal power series solution of the AODE $F(y) = 0$, where $c_i \in \mathbb{K}$ is to be determined. From $[x^0]F(z) = 0$, we have that $c_0^2 - c_0 = 0$.

If we take $c_0 = 1$, then we can deduce from (4) that for each $k \geq 1$,

$$[x^0]F^{(k)}(z) = (k + 1)c_k + \tilde{R}_{k-1}(0, 1, c_1, \ldots, c_{k-1}) = 0.$$

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 1, \ldots, c_{k-1})}{k + 1},$$

where $k \geq 1$. Therefore, we derive a formal power series solution of $F(y) = 0$ with $c_0 = 1$.

[1]This proposition is sometimes called Implicit Function Theorem for AODEs as a folklore.

If we take $c_0 = 0$, then we observe that

$$[x^0]F'(z) = 2c_0c_1 = 0$$

It implies that there is no constraint for c_1 in the equation $[x^0]F'(z) = 0$. For $k \geq 2$, it follows from (4) that

$$[x^0]F^{(k)}(z) = (k-1)c_k + \tilde{R}_{k-1}(0, 0, c_1, \dots, c_{k-1}) = 0.$$

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 0, c_1, \dots, c_{k-1})}{k-1},$$

where $k \geq 2$. Therefore, we derive formal power series solutions of $F(y) = 0$ with $c_0 = 0$ and c_1 is an arbitrary constant in $\mathbb{K}$.

In the above example, we expand $F^{(k)}$ to the second highest derivative $y^{(k)}$ so that there are some non-vanishing coefficients to compute all formal power series solutions of $F(y) = 0$ recursively. In the next sections, we will develop this idea in a systematical way.

3 Refinement of Denef-Lipshitz Lemma

In [DL84][Lemma 2.2], given a differential polynomial F, the authors present an expansion formula for $F^{(2m+2+q)}$ with arbitrary $m, q \in \mathbb{N}$. In this section, by a careful analysis, we refine the expansion and give an explicit formula (see Theorem 3.6). This new formula plays a fundamental role in the computation of formal power series in next sections.

Lemma 3.1. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then we have*

$$\frac{\partial F^{(1)}}{\partial y^{(n+k)}} = \left(\frac{\partial F}{\partial y^{(n+k)}}\right)^{(1)} + \frac{\partial F}{\partial y^{(n+k-1)}}, \tag{5}$$

where $k \leq 1$.

Proof. Assume that $-n \leq k \leq 1$. Then

$$F^{(1)} = \sum_{i=0}^{n} \frac{\partial F}{\partial y^{(i)}} y^{(i+1)} + \frac{\partial F}{\partial x}.$$

Thus,

$$\frac{\partial F^{(1)}}{\partial y^{(n+k)}} = A + B,$$

where

$$A = \sum_{i=0}^{n} \left(\frac{\partial}{\partial y^{(i)}} \left(\frac{\partial F}{\partial y^{(n+k)}} \right) \right) y^{(i+1)} + \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y^{(n+k)}} \right)$$
$$= \left(\frac{\partial F}{\partial y^{(n+k)}} \right)^{(1)},$$

and

$$B = \frac{\partial F}{\partial y^{(n+k-1)}} \cdot \frac{\partial (y^{(n+k)})}{\partial y^{(n+k)}}$$
$$= \frac{\partial F}{\partial y^{(n+k-1)}}.$$

If $k < -n$, then it follow from (2) that both sides of (5) are equal to 0. □

Proposition 3.2. [2] *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, we have*

$$\frac{\partial F^{(m)}}{\partial y^{(n+k)}} = \sum_{j=0}^{m-k} \binom{m}{m-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}} \right)^{(m-k-j)}, \tag{6}$$

where $k \leq m$.

Proof. We use induction on m to prove the above claim.

Let $m = 0$. Then for each $k \leq 0$, the left side of (6) is $\frac{\partial F}{\partial y^{(n+k)}}$. And the right side is

$$\sum_{j=0}^{-k} \binom{0}{-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}} \right)^{(-k-j)} = \frac{\partial F}{\partial y^{(n+k)}}.$$

Assume that the claim holds for m. We consider the $m+1$ case. If $k = m+1$, then the left side of (6) is $\frac{\partial F^{(m+1)}}{\partial y^{(n+m+1)}}$. And the right side is $\frac{\partial F}{\partial y^{(n)}}$. It follows from Lemma 2.1 that the claim holds for $k = m+1$.

If $k \leq m$, the it follows from Lemma 3.1 that

$$\frac{\partial F^{(m+1)}}{\partial y^{(n+k)}} = \frac{\partial \left((F^{(m)})^{(1)} \right)}{\partial y^{(n+k)}} = \left(\frac{\partial F^{(m)}}{\partial y^{(n+k)}} \right)^{(1)} + \frac{\partial F^{(m)}}{\partial y^{(n+k-1)}}.$$

From induction hypothesis, we have that

$$\left(\frac{\partial F^{(m)}}{\partial y^{(n+k)}} \right)^{(1)} = \sum_{j=0}^{m-k} \binom{m}{m-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}} \right)^{(m+1-k-j)}.$$

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[2] We thank Christoph Koutschan for bring this proposition to our attention, which leads to a short proof of Corollary 3.3.

Similarly,

$$\begin{aligned}\frac{\partial F^{(m)}}{\partial y^{(n+k-1)}} &= \sum_{j=0}^{m+1-k} \binom{m}{m+1-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-k-j)} \\ &= \sum_{j=0}^{m-k} \binom{m}{m+1-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-k-j)} + \frac{\partial F}{\partial y^{(n-(m+1-k))}}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\frac{\partial F^{(m+1)}}{\partial y^{(n+k)}} &= \sum_{j=0}^{m-k} \left[\binom{m}{m-k-j} + \binom{m}{m+1-k-j}\right] \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-k-j)} \\ &\quad + \frac{\partial F}{\partial y^{(n-(m+1-k))}} \\ &= \sum_{j=0}^{m-k} \binom{m+1}{m+1-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-k-j)} + \frac{\partial F}{\partial y^{(n-(m+1-k))}} \\ &= \sum_{j=0}^{m+1-k} \binom{m+1}{m+1-k-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-k-j)}.\end{aligned}$$

□

Corollary 3.3. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, we have*

$$\frac{\partial F^{(2m+1)}}{\partial y^{(n+m)}} = \sum_{j=0}^{m+1} \binom{2m+1}{m+1-j} \cdot \left(\frac{\partial F}{\partial y^{(n-j)}}\right)^{(m+1-j)}.$$

Proof. In Proposition 3.2, we set $\tilde{m} = 2m+1$, and $\tilde{k} = m$. □

Proposition 3.4. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, we have*

$$F^{(2m+1)} = \sum_{i=0}^{m} y^{(n+2m+1-i)} \cdot \sum_{j=0}^{i} \binom{2m+1}{j} f_{n+i-j}^{(j)} + r_{n+m},$$

where

1. $f_{n+i} = \frac{\partial F}{\partial y^{(n-i)}}$ *for* $i = 0, \ldots, m$;
2. $r_{n+m} \in \mathbb{K}[x]\{y\}$ *is a differential polynomial of order at most* $n+m$.

Proof. We use induction on m to prove the above claim.

Let $m = 0$. By Lemma 2.1, we have

$$F^{(1)} = S_F \cdot y^{(n+1)} + R_1,$$

where $S_F = \frac{\partial F}{\partial y^{(n)}}$ and $R_1 \in \mathbb{K}[x]\{y\}$ is a differential polynomial of order n. Set $f_n = S_F$ and $r_n = R_1$.

Assume that the claim holds for m. Then

$$\begin{aligned}
F^{(2m+2)} &= \left(F^{(2m+1)}\right)' \\
&= \sum_{i=0}^{m} y^{(n+2m+2-i)} \cdot \sum_{j=0}^{i} \binom{2m+1}{j} f_{n+i-j}^{(j)} \\
&\quad + \sum_{i=0}^{m} y^{(n+2m+1-i)} \cdot \sum_{j=0}^{i} \binom{2m+1}{j} f_{n+i-j}^{(j+1)} + r_{n+m}^{(1)} \\
&= y^{(n+2m+2)} \cdot f_n + \left[\sum_{i=1}^{m} y^{(n+2m+2-i)} \cdot \sum_{j=0}^{i} \binom{2m+1}{j} f_{n+i-j}^{(j)} \right.\\
&\quad \left. + \sum_{i=1}^{m} y^{(n+2m+2-i)} \cdot \sum_{j=0}^{i-1} \binom{2m+1}{j} f_{n+i-j-1}^{(j+1)}\right] \\
&\quad + y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+1)} + r_{n+m}^{(1)} \\
&= \sum_{i=0}^{m} y^{(n+2m+2-i)} \cdot \sum_{j=0}^{i} \binom{2m+2}{j} f_{n+i-j}^{(j)} \\
&\quad + y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+1)} + r_{n+m}^{(1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F^{(2m+3)} &= \left(F^{(2m+2)}\right)' \\
&= \sum_{i=0}^{m} y^{(n+2m+3-i)} \cdot \sum_{j=0}^{i} \binom{2m+2}{j} f_{n+i-j}^{(j)} \\
&\quad + \sum_{i=0}^{m} y^{(n+2m+2-i)} \cdot \sum_{j=0}^{i} \binom{2m+2}{j} f_{n+i-j}^{(j+1)} \\
&\quad + y^{(n+m+2)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+2)} + r_{n+m}^{(2)} \\
&= y^{(n+2m+3)} \cdot f_n \\
&+ \sum_{i=1}^{m} y^{(n+2m+3-i)} \cdot \left[\sum_{j=0}^{i} \binom{2m+2}{j} f_{n+i-j}^{(j)} + \sum_{j=0}^{i-1} \binom{2m+2}{j} f_{n+i-j-1}^{(j+1)} \right] \\
&+ y^{(n+m+2)} \cdot \left[\sum_{j=0}^{m} \binom{2m+2}{j} f_{n+m-j}^{(j+1)} + \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+1)} \right] \\
&+ y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+2)} + r_{n+m}^{(2)} \\
&= \sum_{i=0}^{m} y^{(n+2m+3-i)} \cdot \sum_{j=0}^{i} \binom{2m+3}{j} f_{n+i-j}^{(j)} \\
&+ y^{(n+m+2)} \cdot \sum_{j=0}^{m} \left[\binom{2m+2}{j} + \binom{2m+1}{j} \right] f_{n+m-j}^{(j+1)} \\
&+ y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} f_{n+m-j}^{(j+2)} + r_{n+m}^{(2)}.
\end{aligned}
$$

By Lemma 2.1, we have

$$
f_{n+m-j}^{(j+2)} = \frac{\partial f_{n+m-j}^{(j)}}{\partial y^{(n+m)}} \cdot y^{(n+m+2)} + R_{j,n+m+1}, \tag{7}
$$

$$
r_{n+m}^{(2)} = \frac{\partial r_{n+m}}{\partial y^{(n+m)}} \cdot y^{(n+m+2)} + R_{m+n+1}. \tag{8}
$$

where $R_{j,n+m+1}, R_{n+m+1} \in \mathbb{K}[x]\{y\}$ are of order at most $n+m+1$, $j = 0, \ldots, m$.

Set

$$
\begin{aligned}
f_{n+m+1} = &\sum_{j=0}^{m} \left[\binom{2m+2}{j} + \binom{2m+1}{j} \right] f_{n+m-j}^{(j+1)} - \sum_{j=1}^{m+1} \binom{2m+3}{j} f_{n+m+1-j}^{(j)} \\
&+ y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} \frac{\partial f_{n+m-j}^{(j)}}{\partial y^{(n+m)}} + \frac{\partial r_{n+m}}{\partial y^{(n+m)}}, \\
r_{n+m+1} = &\; y^{(n+m+1)} \cdot \sum_{j=0}^{m} R_{j,n+m+1} + R_{m+n+1}.
\end{aligned}
$$

From the last formula of $F^{(2m+3)}$, (7), (8) and the above two formulas, we have

$$F^{(2m+3)} = \sum_{i=0}^{m+1} y^{(n+2m+3-i)} \cdot \sum_{j=0}^{i} \binom{2m+3}{j} f_{n+i-j}^{(j)} + r_{n+m+1}.$$

Next, we show that $f_{n+m+1} = \frac{\partial F}{\partial y^{(n-m-1)}}$. By the definition of f_{n+m+1}, we see that

$$\begin{aligned} f_{n+m+1} &= \sum_{j=0}^{m} \left[\binom{2m+2}{j} + \binom{2m+1}{j} - \binom{2m+3}{j+1} \right] f_{n+m-j}^{(j+1)} \\ &\quad + y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} \frac{\partial f_{n+m-j}^{(j)}}{\partial y^{(n+m)}} + \frac{\partial r_{n+m}}{\partial y^{(n+m)}} \\ &= y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} \frac{\partial f_{n+m-j}^{(j)}}{\partial y^{(n+m)}} + \frac{\partial r_{n+m}}{\partial y^{(n+m)}} \\ &\quad - \sum_{j=0}^{m} \binom{2m+1}{j+1} f_{n+m-j}^{(j+1)}. \end{aligned}$$

By the induction hypothesis, we have that

$$\begin{aligned} \frac{\partial r_{n+m}}{\partial y^{(n+m)}} &= \frac{\partial}{\partial y^{(n+m)}} \left(F^{(2m+1)} - \sum_{i=0}^{m} y^{(n+2m+1-i)} \cdot \sum_{j=0}^{i} \binom{2m+1}{j} f_{n+i-j}^{(j)} \right) \\ &= \frac{\partial F^{(2m+1)}}{\partial y^{(n+m)}} - y^{(n+m+1)} \cdot \sum_{j=0}^{m} \binom{2m+1}{j} \frac{\partial f_{n+m-j}^{(j)}}{\partial y^{(n+m)}}. \end{aligned}$$

Using the above formula, it follows that

$$f_{n+m+1} = \frac{\partial F^{(2m+1)}}{\partial y^{(n+m)}} - \sum_{j=0}^{m} \binom{2m+1}{j+1} f_{n+m-j}^{(j+1)}.$$

By the induction hypothesis, we have

$$\begin{aligned} f_{n+m+1} &= \frac{\partial F^{(2m+1)}}{\partial y^{(n+m)}} - \sum_{j=0}^{m} \binom{2m+1}{j+1} \left(\frac{\partial F}{\partial y^{(n-m+j)}} \right)^{(j+1)} \\ &= \frac{\partial F^{(2m+1)}}{\partial y^{(n+m)}} - \sum_{j=0}^{m} \binom{2m+1}{m+1-j} \left(\frac{\partial F}{\partial y^{(n-j)}} \right)^{(m+1-j)}. \end{aligned}$$

By Corollary 3.3, we conclude that

$$f_{n+m+1} = \frac{\partial F}{\partial y^{(n-m-1)}}.$$

□

The following theorem is the main result of this section. This theorem can be considered as a refinement of [DL84, Lemma 2.2].

Theorem 3.5. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m, k \in \mathbb{N}$, we have*

$$F^{(2m+1+k)} = \sum_{i=0}^{m} y^{(n+2m+1+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+1+k}{j} f_{n+i-j}^{(j)} + r_{n+m+k},$$

where

1. $f_{n+i} = \frac{\partial F}{\partial y^{(n-i)}}$ *for* $i = 0, \ldots, m$;
2. $r_{n+m+k} \in \mathbb{K}[x]\{y\}$ *is a differential polynomial of order at most* $n + m + k$.

Proof. We use induction on k to prove the above claim.

Let $k = 0$. It follows from Proposition 3.4.

Assume that the claim holds for k. Then we have

$$\begin{aligned} F^{(2m+2+k)} &= \left(F^{(2m+1+k)}\right)' \\ &= \sum_{i=0}^{m} y^{(n+2m+2+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+1+k}{j} f_{n+i-j}^{(j)} \\ &\quad + \sum_{i=0}^{m} y^{(n+2m+1+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+1+k}{j} f_{n+i-j}^{(j+1)} + r_{n+m+k}^{(1)} \\ &= y^{(n+2m+2+k)} \cdot f_n + \left[\sum_{i=1}^{m} y^{(n+2m+2+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+1+k}{j} f_{n+i-j}^{(j)} \right. \\ &\quad \left. + \sum_{i=1}^{m} y^{(n+2m+2+k-i)} \cdot \sum_{j=0}^{i-1} \binom{2m+1+k}{j} f_{n+i-j-1}^{(j+1)} \right] \\ &\quad + y^{(n+m+k+1)} \cdot \sum_{j=0}^{m} \binom{2m+1+k}{j} f_{n+m-j}^{(j+1)} + r_{n+m+k}^{(1)} \\ &= \sum_{i=0}^{m} y^{(n+2m+2+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+2+k}{j} f_{n+i-j}^{(j)} \\ &\quad + y^{(n+m+k+1)} \cdot \sum_{j=0}^{m} \binom{2m+1+k}{j} f_{n+m-j}^{(j+1)} + r_{n+m+k}^{(1)}. \end{aligned}$$

Set $r_{n+m+k+1} = y^{(n+m+k+1)} \cdot \sum_{j=0}^{m} \binom{2m+1+k}{j} f_{n+m-j}^{(j+1)} + r_{n+m+k}^{(1)}$, which is a differential polynomial of order at most $n+m+k+1$. Then

$$F^{(2m+2+k)} = \sum_{i=0}^{m} y^{(n+2m+2+k-i)} \cdot \sum_{j=0}^{i} \binom{2m+2+k}{j} f_{n+i-j}^{(j)} + r_{n+m+k+1}.$$

□

Assume that $F \in \mathbb{K}[x]\{y\}$ is a differential polynomial of order n. For each $m, k \in \mathbb{N}$, we define

$$\mathcal{B}_m(k) = \begin{bmatrix} \binom{2m+1+k}{0} & \binom{2m+1+k}{1} & \cdots & \binom{2m+1+k}{m} \end{bmatrix},$$

and

$$\mathcal{S}_m(F) = \begin{bmatrix} \frac{\partial F}{\partial y^{(n)}} & \frac{\partial F}{\partial y^{(n-1)}} & \frac{\partial F}{\partial y^{(n-2)}} & \cdots & \frac{\partial F}{\partial y^{(n-m)}} \\ 0 & \left(\frac{\partial F}{\partial y^{(n)}}\right)^{(1)} & \left(\frac{\partial F}{\partial y^{(n-1)}}\right)^{(1)} & \cdots & \left(\frac{\partial F}{\partial y^{(n-m-1)}}\right)^{(1)} \\ 0 & 0 & \left(\frac{\partial F}{\partial y^{(n)}}\right)^{(2)} & \cdots & \left(\frac{\partial F}{\partial y^{(n-m-2)}}\right)^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \left(\frac{\partial F}{\partial y^{(n)}}\right)^{(m)} \end{bmatrix},$$

and

$$Y_m = \begin{bmatrix} y^{(m)} \\ y^{(m-1)} \\ \cdots \\ y \end{bmatrix}.$$

Then we can write Theorem 3.5 into the following matrix form:

Theorem 3.6. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m, k \in \mathbb{N}$, we have*

$$F^{(2m+1+k)} = \mathcal{B}_m(k) \cdot \mathcal{S}_m(F) \cdot Y_m^{(n+m+k+1)} + r_{n+m+k}, \tag{9}$$

where $r_{n+m+k} \in \mathbb{K}[x]\{y\}$ is a differential polynomial of order at most $n+m+k$.

Proof. It follows from Theorem 3.5. □

Definition 3.7. *Assume that $F \in \mathbb{K}[x]\{y\}$ is a differential polynomial F of order $n \in \mathbb{N}$. For $m \in \mathbb{N}$, we call $\mathcal{S}_m(F)$ the m-th* separant matrix *of F.*

Note that the 0-th separant matrix is exactly the usual separant $\frac{\partial F}{\partial y^{(n)}}$ of F.

4 Non-vanishing properties at an initial tuple

In this section, we consider the problem of deciding when a solution modulo a certain power of x of a given AODE can be extended to a full formal power series solution. As a nice application of Theorem 3.6, we present a partial answer for this problem. In particular, given a certain number of coefficients satisfying some additional assumptions, we propose an algorithm to check whether there is a formal power series solution whose first coefficients are the given ones, and in the affirmative case, compute all of them (see Theorem 4.5 and Algorithm 4.6).

Let $m, n \in \mathbb{N}$ and $m \geq n$. As a matter of notation, we set

$$\begin{array}{rccc} \pi_n : & \mathbb{K}^{m+1} & \longrightarrow & \mathbb{K}^{n+1} \\ & (c_0, \ldots, c_m) & \longmapsto & (c_0, \ldots, c_n). \end{array}$$

We call π_n the n-th *natural projection map* from $\mathbb{K}^{m+1}$ to $\mathbb{K}^{n+1}$.

Definition 4.1. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$.*

1. *Assume that $\boldsymbol{c} = (c_0, c_1, \ldots)$ is a tuple of indeterminates and $m \in \mathbb{N}$. We call $\mathcal{J}_m(F) = \langle F(\boldsymbol{c}), \ldots, F^{(m)}(\boldsymbol{c})\rangle \subseteq \mathbb{K}[c_0, \ldots, c_{n+m}]$ the m-th* jet ideal *of F. We denote the zero points of $J_m(F)$ by $\mathcal{Z}(\mathcal{J}_m(F))$.*

2. *Let $k \in \mathbb{N}$. Assume that $\boldsymbol{c} = (c_0, c_1, \ldots, c_k) \in \mathbb{K}^{k+1}$. We say that $\boldsymbol{c}$ can be extended to a formal power series solutions of $F(y) = 0$ if there exists $z \in \mathbb{K}[[x]]$ such that $F(z) = 0$ and*

$$z \equiv c_0 + \frac{c_1}{1!}x + \cdots + \frac{c_k}{k!}x^k \mod x^{k+1}.$$

3. *Let $k, m \in \mathbb{N}$ with $k \leq m$. Assume that $\boldsymbol{c} \in \mathbb{K}^{k+1}$. We say that $\boldsymbol{c}$ can be extended to a zero point of $\mathcal{J}_m(F)$ if there exists $\tilde{\boldsymbol{c}} \in \mathcal{Z}(\mathcal{J}_m(F))$ such that $\pi_k(\tilde{\boldsymbol{c}}) = \boldsymbol{c}$.*

Lemma 4.2. *Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Set $\boldsymbol{c} = (c_0, c_1, \ldots)$ to be a tuple of indeterminates. Assume that the j-th separant matrix $\mathcal{S}_j(F)(\boldsymbol{c}) = 0$ for all $j < m$. Then*

1. *for each $0 \leq k \leq 2m$, the polynomial $F^{(k)}(\boldsymbol{c})$ involves only $c_0, \ldots, c_{n+m}$.*

2. *for each $k \geq 2m$, the polynomial $F^{(k)}(\boldsymbol{c})$ involves only $c_0, \ldots, c_{n+k-m}$.*

Proof. 1. The claim is clear for $0 \leq k \leq m$. Assume that $m < k \leq 2m$. Set

$$\tilde{m} = k - 1 - m \text{ and } \tilde{k} = 2m - k + 1.$$

Then it is straightforward to verify that

$$0 \leq \tilde{m} \leq m-1, \quad 1 \leq \tilde{k} \leq m, \quad \text{and} \quad \tilde{m}+\tilde{k}=m.$$

By Theorem 3.6, we have

$$\begin{aligned} F^{(k)}(\mathbf{c}) &= F^{(2\tilde{m}+1+\tilde{k})}(\mathbf{c}) \\ &= \mathcal{B}_{\tilde{m}}(\tilde{k}) \cdot \mathcal{S}_{\tilde{m}}(F)(\mathbf{c}) \cdot \begin{pmatrix} c_{n+2\tilde{m}+\tilde{k}+1} \\ \cdots \\ c_{n+\tilde{m}+\tilde{k}+1} \end{pmatrix} + r_{n+\tilde{m}+\tilde{k}}(\mathbf{c}), \end{aligned} \tag{10}$$

where $r_{n+\tilde{m}+\tilde{k}}$ is a differential polynomial of order at most $n+m$. On account of $0 \leq \tilde{m} \leq m-1$, we have $\mathcal{S}_{\tilde{m}}(F)(\mathbf{c}) = 0$. Therefore, it follows from (10) that

$$F^{(k)}(\mathbf{c}) = r_{n+\tilde{m}+\tilde{k}}(\mathbf{c}),$$

which only involves $c_0, \ldots, c_{n+m}$.

2. The proof is similar to that of the above item by setting $k = 2\tilde{m}+1+\tilde{k}$ with $\tilde{m} = m-1$ and $\tilde{k} = k-2m+1$.

□

Definition 4.3. *Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ a differential polynomial of order n. Let $\mathbf{c} = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$. We say that F is* m-th non-vanishing at $\mathbf{c}$ *if the following conditions hold:*

1. *$\mathcal{S}_k(F)(\mathbf{c}) = 0$ for all $k < m$, and $\mathcal{S}_m(F)(\mathbf{c}) \neq 0$,*
2. *$F(\mathbf{c}) = \cdots = F^{(2m)}(\mathbf{c}) = 0$.*

As a consequence of Lemma 4.2, item 1 of the above definition implies that the identities in item 2 are well-defined.

Remark 4.4. *Recall that a solution of an AODE $F(y) = 0$ of order n is called* non-singular *if it does not vanish the separant $\frac{\partial F}{\partial y^{(n)}}$. One can check that if a formal power series $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is a non-singular solution of $F(y) = 0$, then there exists an $m \in \mathbb{N}$ such that F is m-th non-vanishing at $(c_0, \ldots, c_{n+m})$.*

Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\mathbf{c} = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$ and F is m-th non-vanishing at $\mathbf{c}$. We denote:

$$\mathbf{g}_{F,\mathbf{c}}(t) = \sum_{i=0}^{m} \binom{2m+1+t}{i} \left(\frac{\partial F}{\partial y^{(n-m+i)}} \right)^{(i)} (\mathbf{c}) \in \mathbb{K}[t],$$

$$\mathbf{r}_{F,\mathbf{c}} = \text{the number of non-negative integer roots of } \mathbf{g}_{F,\mathbf{c}}(t).$$

Furthermore, we set:

$$\mathbf{q}_{F,\mathbf{c}} = \begin{cases} \text{the largest integer root of } \mathbf{g}_{F,\mathbf{c}}, & \text{if } \mathbf{r}_{F,\mathbf{c}} \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.5. *Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\boldsymbol{c} \in \mathbb{K}^{n+m+1}$ and F is m-th non-vanishing at $\boldsymbol{c}$. Then*

1. *The tuple $\boldsymbol{c}$ can be extended to a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero of $\mathcal{J}_{2m+\mathbf{q}_{F,\mathbf{c}}+1}(F)$.*

2. *Set*

$$\mathcal{V}_{\boldsymbol{c}}(F) = \pi_{n+m+\mathbf{q}_{F,\mathbf{c}}+1}\left(\left\{\tilde{\mathbf{c}} \in \mathcal{Z}(\mathcal{J}_{2m+\mathbf{q}_{F,\mathbf{c}}+1}(F))\,|\pi_{n+m}(\tilde{\mathbf{c}}) = \boldsymbol{c}\right\}\right).$$

Then $\mathcal{V}_{\boldsymbol{c}}(F)$ [3] is an affine variety of dimension at most $\mathbf{r}_{F,c}$. Moreover, each point of it can be extended uniquely to a formal power series solution of $F(y) = 0$.

Proof. 1. Let $\tilde{\mathbf{c}} = (\tilde{c_0}, \tilde{c}_1, \ldots) \in \mathbb{K}^{\mathbb{N}}$ such that $\pi_{n+m}(\tilde{\mathbf{c}}) = \mathbf{c}$, where $\tilde{c}_{n+m+k}$ is to be determined for $k > 0$. Assume that $z = \sum_{i=0}^{\infty} \frac{\tilde{c}_i}{i!} x^i$ is a formal power series solution of $F(y) = 0$.

Necessity: It follows from the remark before Proposition 2.3.

Sufficiency: Since $\mathbf{c}$ is a zero point of $\mathcal{J}_{2m}(F)$, we conclude from the remark before Proposition 2.3 that z is a solution of $F(y) = 0$ if and only if for each $k \in \mathbb{N}$, we have

$$F^{(2m+1+k)}(\tilde{\mathbf{c}}) = 0. \tag{11}$$

On the other hand, since $\mathcal{S}_m(F)(\tilde{\mathbf{c}}) = \mathcal{S}_m(F)(\mathbf{c})$, we have

$$\mathcal{B}_m(t) \cdot \mathcal{S}_m(F)(\tilde{\mathbf{c}}) = [0, \ldots, 0, \mathbf{g}_{F,\mathbf{c}}(t)] \in \mathbb{K}[t]^{m+1}.$$

As a consequence of Theorem 3.6, the equation (11) can be written as

$$F^{(2m+1+k)}(\tilde{\mathbf{c}}) = \mathbf{g}_{F,\mathbf{c}}(k) \cdot \tilde{c}_{n+m+k+1} + r_{n+m+k}(\tilde{c}_0, \ldots, \tilde{c}_{n+m+k}) = 0, \tag{12}$$

[3] For $k = 0, \ldots, \mathbf{q}_{F,\mathbf{c}}$, the equations (12) only contain $c_0, \ldots, c_{n+m+\mathbf{q}_{F,\mathbf{c}}+1}$, and has no constraints for $c_{n+m+\mathbf{q}_{F,\mathbf{c}}+2}, \ldots, c_{n+2m+\mathbf{q}_{F,\mathbf{c}}+1}$. Thus, in the definition of $\mathcal{V}_{\mathbf{c}}(F)$, we take the $(n + m + \mathbf{q}_{F,\mathbf{c}} + 1)$-th projection map to the zero set of $\mathcal{J}_{n+2m+\mathbf{q}_{F,\mathbf{c}}+1}(F)$.

where r_{n+m+k} is a differential polynomial of order at most $n+m+k$. Since $\mathbf{c}$ can be extended to a zero point of $\mathcal{J}_{2m+\mathbf{q}_{F,\mathbf{c}}+1}(F)$, there exists $\tilde{c}_{n+m+1}, \ldots, \tilde{c}_{n+m+\mathbf{q}_{F,\mathbf{c}}+1}$ in $\mathbb{K}$ such that (12) holds for $k = 0, \ldots, \mathbf{q}_{F,\mathbf{c}}$. For $k > \mathbf{q}_{F,\mathbf{c}}$, we set

$$\tilde{c}_{n+m+k+1} = -\frac{r_{n+i+k}(0, \tilde{c}_0, \ldots, \tilde{c}_{n+i+k})}{\mathbf{g}_{F,\mathbf{c}}(k)}. \tag{13}$$

Then z is a solution of $F(y) = 0$.

2. Assume that $\mathbf{c} = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$. By (12) and the definition of $\mathcal{V}_{\mathbf{c}}(F)$, we see that $\mathcal{V}_{\mathbf{c}}(F)$ is the set of zero points of the following equations:

1. $\tilde{c}_k = c_k \quad$ for $0 \leq k \leq n+m$;
2. $\mathbf{g}_{F,\mathbf{c}}(k) \cdot \tilde{c}_{n+m+k+1} + r_{n+m+k}(\tilde{c}_0, \ldots, \tilde{c}_{n+m+k}) = 0 \quad$ for $0 \leq k \leq \mathbf{q}_{F,\mathbf{c}}$,

where $\tilde{c}_k$'s are indeterminates, $0 \leq k \leq n+m+\mathbf{q}_{F,\mathbf{c}}+1$. Thus, $\mathcal{V}_{\mathbf{c}}(F)$ is an affine variety.

If $\mathbf{r}_{F,\mathbf{c}} = 0$, then it is straightforward to verify that $\mathcal{V}_{\mathbf{c}}(F)$ has at most one point. Therefore, it has at most dimension zero.

Assume that $\mathbf{r}_{F,\mathbf{c}} \geq 1$. Let $k_1 < \cdots < k_{\mathbf{r}_{F,\mathbf{c}}} = \mathbf{q}_{F,\mathbf{c}}$ be positive integer roots of $\mathbf{g}_{F,\mathbf{c}}(t)$. If $k \notin \{k_1, \ldots, k_{\mathbf{r}_{F,\mathbf{c}}}\}$, then it follows from (12) that $\tilde{c}_{n+m+k+1}$ is uniquely determined from the previous coefficients. Then we see that

$$\begin{array}{rccc} \phi: & \mathcal{V}_{\mathbf{c}}(F) & \longrightarrow & \mathbb{K}^{\mathbf{r}_{F,\mathbf{c}}} \\ & \tilde{\mathbf{c}} & \longmapsto & (\tilde{c}_{n+m+k_1+1}, \ldots, \tilde{c}_{n+m+k_{\mathbf{r}_{F,\mathbf{c}}}+1}) \end{array}$$

is an injective map. Therefore, we conclude that $\mathcal{V}_{\mathbf{c}}(F)$ is of dimension at most $\mathbf{r}_{F,\mathbf{c}}$. Moreover, it follows from (13) that each point of $\mathcal{V}_{\mathbf{c}}(F)$ can be uniquely extended to a formal power series solution of $F(y) = 0$.

□

Note that in the above theorem, the affine variety $\mathcal{V}_{\mathbf{c}}(F)$ is in bijection with the set of all formal power series solutions of $F(y) = 0$ whose initial $(n+m+1)$-tuple is $\mathbf{c}$.

The proof of the above theorem is constructive. If a tuple $\mathbf{c} \in \mathbb{K}^{n+m+1}$ satisfies the condition that F is m-th non-vanishing at $\mathbf{c}$, then the proof gives an algorithm to decide whether $\mathbf{c}$ can be extended to a formal power series solution of $F(y) = 0$ or not, and in the affirmative case determine all of them. We summarize them as the following algorithm.

Algorithm 4.6. *Given $\ell \in \mathbb{N}$, $\mathbf{c} \in \mathbb{K}^{n+m+1}$, and a differential polynomial F of order n which is m-th non-vanishing at $\mathbf{c}$, decide whether $\mathbf{c}$ can be extended to a formal power series solution of $F(y) = 0$ or not. In the affirmative case, compute a truncated power series solutions of $F(y) = 0$ up to degree ℓ and* $\dim(\mathcal{V}_{\mathbf{c}}(F))$.

1. *Compute* $\boldsymbol{g}_{F,\mathbf{c}}$, $\boldsymbol{r}_{F,\mathbf{c}}$, $\boldsymbol{q}_{F,\mathbf{c}}$ *and the following defining equations of* $\mathcal{V}_{\mathbf{c}}(F)$ *by Theorem 3.6:*

   (a) $\tilde{c}_k = c_k \quad$ *for* $0 \leq k \leq n+m$;

   (b) $\boldsymbol{g}_{F,\mathbf{c}}(k) \cdot \tilde{c}_{n+m+k+1} + r_{n+m+k}(\tilde{c}_0, \ldots, \tilde{c}_{n+m+k}) = 0 \quad$ *for* $0 \leq k \leq \boldsymbol{q}_{F,\mathbf{c}}$,

   where $\tilde{c}_k$*'s are indeterminates,* $0 \leq k \leq n+m+\boldsymbol{q}_{F,\mathbf{c}}+1$.

2. *Check whether* $\mathcal{V}_{\mathbf{c}}(F)$ *is empty or not by using Gröbner bases. If* $\mathcal{V}_{\mathbf{c}}(F)$ *is an empty set, then output the string "***c** *can not be extended to a formal power series solution of* $F(y)=0$*". Otherwise, go to the next step.*

3. *Compute one point* $\tilde{\mathbf{c}} \in \mathbb{K}^{n+m+\boldsymbol{q}_{F,\mathbf{c}}+2}$ *of* $\mathcal{V}_{\mathbf{c}}(F)$ *by using Gröbner bases. For* $n+m+\boldsymbol{q}_{F,\mathbf{c}}+1 < i \leq \ell$, *compute* c_i *by using* (13).

4. *Return* $\sum_{i=0}^{\ell} \frac{c_i}{i!} x^i$ *and* $\dim(\mathcal{V}_{\mathbf{c}}(F))$ *[KW88].*

The termination of the above algorithm is evident. The correctness follows from Theorem 4.5.

Example 4.7. Consider the following AODE of order 2:

$$F = xy'' - 3y' + x^2y^2 = 0.$$

Let $\mathbf{c} = (c_0, 0, 0, 2c_0^2) \in \pi_3(\mathcal{Z}(\mathcal{J}_2(F)))$, where c_0 is an arbitrary constant in $\mathbb{K}$. One can verify that each point of $\pi_3(\mathcal{Z}(\mathcal{J}_2(F)))$ is in the form of $\mathbf{c}$. A direct calculation implies that F is first non-vanishing at $\mathbf{c}$.

Moreover, we have that $\mathbf{g}_{F,\mathbf{c}} = t$ and $\mathbf{q}_{F,\mathbf{c}} = 0$. Thus, it follows that

$$\mathcal{V}_{\mathbf{c}}(F) = \{\tilde{\mathbf{c}} = (c_0, 0, 0, 2c_0^2, c_4) \in \mathbb{K}^5 \mid c_4 \in \mathbb{K}\}.$$

So, the dimension of $\mathcal{V}_{\mathbf{c}}(F)$ is equal to one and the corresponding formal power series solutions are

$$z \equiv c_0 + \frac{c_0^2}{3}x^3 + \frac{c_4}{24}x^4 - \frac{c_0^3}{18}x^6 - \frac{c_0c_4}{252}x^7 - \frac{c_0^2c_4}{3024}x^{10} \mod x^{11}.$$

Above all, the set of formal power series solutions of $F(y)=0$ at the origin is in bijection with $\mathbb{K}^2$.

5 Non-vanishing differential polynomials

The input specification of Algorithm 4.6 is that the given initial tuple $\mathbf{c}$ is of length $n+m+1$ and that the differential polynomial F is m-th non-vanishing at $\mathbf{c}$. In general, the natural numbers m can be arbitrary large. In this section, we give a sufficient condition for a differential polynomial in which the existence for an upper bound of m is guaranteed. Differential polynomial satisfying the sufficient condition will be called *non-vanishing*. For non-vanishing differential polynomials, we give a complete algorithm for deciding whether a solution modulo a certain power of x can be extended to a formal power series solution, and in the affirmative case, compute all of them (see Theorem 5.6 and Algorithm 5.7). Statistical investigation in Section 6 shows us that non-vanishing differential polynomials form a big class among algebraic differential polynomials.

Definition 5.1. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Assume that $\boldsymbol{c} = (c_0, c_1, \ldots)$ is a tuple of indeterminates and $m \in \mathbb{N}$.*

1. *We define $\mathcal{I}_m(F)$ to be the ideal of $\mathbb{K}[c_0, \ldots, c_{n+m}]$ generated by the entries of the matrix $\mathcal{S}_m(F)(\boldsymbol{c})$.*

2. *The differential polynomial F is called* m-th non-vanishing *if m is the smallest natural number such that $1 \in \langle \mathcal{I}_m(F), \mathcal{J}_{2m}(F)\rangle_{\mathbb{K}[c_0,\ldots,c_{n+2m}]}$, where $\mathcal{J}_{2m}(F)$ is the $2m$-th jet ideal of F. We call m the* non-vanishing number *of F.*

3. *We call F a* non-vanishing *differential polynomial if there exists $m \in \mathbb{N}$ such that F is m-th non-vanishing. Otherwise, we call F a* vanishing *differential polynomial.*

Example 5.2. Consider the AODE in Example 2.4:

$$F = xy' + y^2 - y - x^2 = 0.$$

By computation, we find that $1 \notin \langle c_0^2 - c_0\rangle_{\mathbb{K}[c_0]} = \langle \mathcal{I}_0, \mathcal{J}_0\rangle_{\mathbb{K}[c_0]}$. Furthermore, we have $1 \in \langle \mathcal{I}_1, \mathcal{J}_2\rangle_{\mathbb{K}[c_0,c_1]}$ because $(S_F)' = 1$. Therefore, F is a first non-vanishing differential polynomial.

Proposition 5.3. *Every irreducible differential polynomial of order zero is non-vanishing.*

Proof. Assume that $A(x, y)$ is an irreducible polynomial in $\mathbb{K}[x, y]$. Set

$$S_A = \frac{\partial A}{\partial y} \in \mathbb{K}[x, y].$$

Since $A(x,y)$ is irreducible in $\mathbb{K}[x,y]$, we have that $A(x,y)$ is also irreducible in $\mathbb{K}(x)[y]$. Therefore, $\gcd(A, S_A) = 1$ in $\mathbb{K}(x)[y]$. By the Bézout's identity, there exist $U, V \in \mathbb{K}(x)[y]$ such that

$$UA + VS_A = 1.$$

By clearing the denominators of the above equation, we know that there exist $\tilde{U}, \tilde{V} \in \mathbb{K}[x,y]$ and $a \in \mathbb{K}[x]\backslash\{0\}$ such that

$$\tilde{U}A + \tilde{V}S_A = a.$$

Let $d = \deg_x(a)$. Differentiating both sides of the above equation for d times, we have that

$$\sum_{i=0}^{d} \binom{d}{i} \tilde{U}^{(i)} A^{(d-i)} + \sum_{i=0}^{d} \binom{d}{i} \tilde{V}^{(i)} S_A^{(d-i)} = c,$$

where c is a nonzero constant in $\mathbb{K}$. It implies that

$$1 \in \langle \mathcal{I}_d(A), \mathcal{J}_{2d}(A) \rangle.$$

Therefore, we conclude from Definition 5.1 that A is at most a d-th nonvanishing differential polynomial. □

Below is an example for vanishing differential polynomials.

Example 5.4. Consider the following AODE :

$$F = (y')^2 + y^3 = 0.$$

A direct computation implies that for each $m \in \mathbb{N}$, we have

$$\mathcal{I}_m(F) + \mathcal{J}_{2m}(F) \subseteq \langle c_0, c_1, \ldots, c_{2m+1} \rangle.$$

Therefore, it follows from item 3 of Definition 5.1 that F is a vanishing differential polynomial.

Next, we show that for each $m \in \mathbb{N}$, there exists an m-th non-vanishing AODE.

Example 5.5. Assume that $m \in \mathbb{N}$. Consider the following AODE:

$$F = \frac{(y' + y)^2}{2} + x^{2m} = 0.$$

For $m = 0$, it is straightforward to see that F is a 0-th nonvanishing differential polynomial.

Let $m > 0$. By computation, we find that $\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y} = y' + y$. Therefore, we have that

$$\mathcal{I}_m(F) = \langle c_1 + c_0, \ldots, c_{m+1} + c_m \rangle. \tag{14}$$

For each $k \geq 0$, it is straightforward to see that $((y' + y)^2)^{(k)}$ is a $\mathbb{K}$-linear combination of terms of the form $(y^{(i)} + y^{(i+1)})(y^{(j)} + y^{(j+1)})$ with $i + j = k$, and $i, j \geq 0$. Therefore, we conclude that for each $0 \leq k \leq m-1$, the jet ideal $\mathcal{J}_{2k}(F)$ is contained in $\mathcal{I}_m(F)$. It implies that

$$\mathcal{I}_k(F) + \mathcal{J}_{2k}(F) \subseteq \mathcal{I}_m(F) \quad \text{for } 0 \leq k \leq m-1.$$

By (14) and the above formula, we have

$$1 \notin \mathcal{J}_{2k}(F) + \mathcal{I}_k(F) \quad \text{for } 0 \leq k \leq m-1.$$

Furthermore, we have that

$$F^{(2m)}(0, c_1, \ldots, c_{1+2m}) \equiv (2m)! \mod \mathcal{I}_m(F).$$

Thus, it follows that

$$1 \in \mathcal{I}_m(F) + \mathcal{J}_{2m}(F).$$

By item 2 of Definition 5.1, we conclude that F is an m-th non-vanishing differential polynomial.

The following theorem is a generalization of Proposition 2.3.

Theorem 5.6. *Let $m, n \in \mathbb{N}$. Assume that F is an m-th non-vanishing differential polynomial of order n and $\boldsymbol{c} \in \mathcal{Z}(\mathcal{J}_{2m}(F))$. Then*

1. *There exists $i \in \{0, \ldots, m\}$ such that F is i-th non-vanishing at $\tilde{\mathbf{c}} = \pi_{n+i}(\boldsymbol{c})$.*

2. *Set $M = \max\{2m + i, 2i + \boldsymbol{q}_{F,\tilde{\mathbf{c}}} + 1\}$. Then $\boldsymbol{c}$ can be extended to a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero point of $\mathcal{J}_M(F)$.*

3. *Set*

$$\mathcal{Z}_{\boldsymbol{c}}(F) = \pi_{n+M-i}\left(\{\hat{\mathbf{c}} \in \mathcal{Z}(\mathcal{J}_M(F)) \,|\, \pi_{n+2m}(\hat{\mathbf{c}}) = \boldsymbol{c}\}\right).$$

   Then $\mathcal{Z}_{\boldsymbol{c}}(F)$ is an affine variety of dimension at most $\boldsymbol{r}_{F,\tilde{\mathbf{c}}}$. Moreover, each point of it can be uniquely extended to a formal power series solution of $F(y) = 0$.

Proof. 1. Since $\mathbf{c} \in \mathcal{Z}(\mathcal{J}_{2m}(F))$ and F is an m-th non-vanishing differential polynomial, it follows that there exists a minimal $i \in \{0, \ldots, m\}$ such that

$$\mathcal{S}_i(F)(\tilde{\mathbf{c}}) = \mathcal{S}_i(F)(\mathbf{c}) \neq 0.$$

By item 2 of Lemma 4.2, we have $F^{(k)}(\tilde{\mathbf{c}}) = 0$ for $k = 0, \ldots, 2i$. Thus, the differential polynomial F is i-th non-vanishing at $\tilde{\mathbf{c}}$.

2 and 3. The proofs are literally the same as that of Theorem 4.5. □

Note that in the above theorem, the affine variety $\mathcal{Z}_{\mathbf{c}}(F)$ is in bijection with the set of all formal power series solutions of $F(y) = 0$ whose initial $(n+2m+1)$-tuple is $\mathbf{c}$.

As a consequence of the above theorem, the set of formal power series solutions of $F(y) = 0$ at the origin is in bijection with the set

$$\bigcup_{\mathbf{c} \in \mathcal{Z}(\mathcal{J}_{2m}(F))} \mathcal{Z}_{\mathbf{c}}(F).$$

Moreover, the proof of the above theorem is constructive. Actually, it gives rise to an algorithm for deciding whether an initial $(n+2m+1)$-tuple $\mathbf{c}$ can be extended to a formal power series solution of $F(y) = 0$ or not. In the affirmative case, it also gives a method to compute a formal power series solution z of $F(y) = 0$ such that

$$z \equiv c_0 + \frac{c_1}{1!}x + \cdots + \frac{c_{n+2m}}{(n+2m)!}x^{n+2m} \mod x^{n+2m+1}.$$

We summarize them as the following algorithm.

Algorithm 5.7. *Given an m-th non-vanishing differential polynomial F of order n, $\boldsymbol{c} \in \mathcal{Z}(\mathcal{J}_{2m}(F))$ and $\ell \in \mathbb{N}$, decide whether $\boldsymbol{c}$ can be extended to be a formal power series solution of F or not. In the affirmative case, compute a truncated power series solutions of F up to degree ℓ and* $\dim(\mathcal{Z}_{\boldsymbol{c}}(F))$.

1. *Determine $i \in \{0, \ldots, m\}$ such that F is i-th non-vanishing at $\tilde{\boldsymbol{c}} = \pi_{n+i}(\boldsymbol{c})$.*

2. *Compute $\boldsymbol{g}_{F,\tilde{\mathbf{c}}}$, $\boldsymbol{r}_{F,\tilde{\mathbf{c}}}$, $\boldsymbol{q}_{F,\tilde{\mathbf{c}}}$ and the following defining equations of $\mathcal{Z}_{\boldsymbol{c}}(F)$ by Theorem 3.6:*

   (a) $\tilde{c}_k = c_k$ *for* $0 \leq k \leq n+2m$;

   (b) $\boldsymbol{g}_{F,\tilde{\mathbf{c}}}(k) \cdot \tilde{c}_{n+i+k+1} + r_{n+i+k}(\tilde{c}_0, \ldots, \tilde{c}_{n+i+k}) = 0$ *for* $2m - 2i \leq k \leq M - 2i - 1$,

   where $\tilde{c}_k$'s are indeterminates, $0 \leq k \leq n + M - i$, and r_{n+i+k} is specified as that in Theorem 3.6.

3. *Check whether* $\mathcal{Z}_{\mathbf{c}}(F)$ *is empty or not by using Gröbner bases. If* $\mathcal{Z}_{\mathbf{c}}(F)$ *is an empty set, then output the string "*$\mathbf{c}$ *can not be extended to a formal power series solution of* $F(y)=0$*". Otherwise, go to the next step.*

4. *Compute one point* $\hat{\mathbf{c}} = (\hat{c}_0, \ldots, \hat{c}_{n+M-i}) \in \mathbb{K}^{n+M-i+1}$ *of* $\mathcal{Z}_{\mathbf{c}}(F)$ *by using Gröbner bases. For* $n+M-i < j \leq \ell$*, set*

$$\hat{c}_j = -\frac{r_{j-1}(\hat{c}_0, \ldots, \hat{c}_{j-1})}{\mathbf{g}_{F,\tilde{\mathbf{c}}}(j-n-i-1)},$$

where r_{j-1} *is specified as that in Theorem 3.6.*

5. *Return* $\sum_{i=0}^{\ell} \frac{\hat{c}_i}{i!} x^i$ *and* $\dim(\mathcal{Z}_{\mathbf{c}}(F))$*.*

The termination of the above algorithm is obvious. The correctness follows from Theorem 5.6.

In [DL84], Lemma 2.3 only concerns non-singular formal power series solutions of a given AODE. Our method also can be used to find singular solutions of a non-vanishing AODE, as the following example illustrates.

Example 5.8. Consider the following AODE:

$$F = y'^2 + y' - 2y - x = 0.$$

By computation, we find that F is a first non-vanishing differential polynomial.

Let $\mathbf{c} = (-\frac{1}{8}, -\frac{1}{2}, 0, c_3)$, where c_3 is an arbitrary constant in $\mathbb{K}$. It is straightforward to verify that $\mathbf{c}$ is a zero point of $\mathcal{J}_2(F)$. Furthermore, we have that F is first non-vanishing at $\tilde{\mathbf{c}} = \pi_2(\mathbf{c}) = (-\frac{1}{8}, -\frac{1}{2}, 0)$.

We find that $\mathbf{g}_{F,\tilde{\mathbf{c}}} = -2$ and $M = 3$. From item 2 of Theorem 5.6, we know that $\mathbf{c}$ can be extended into a formal power solution of $F(y) = 0$ if and only it can be extended to a zero point of $\mathcal{J}_3(F)$.

By calculation, we see that $\mathbf{c}$ can be extend to a zero point of $\mathcal{J}_3(F)$ if and only if $c_3 = 0$. In the affirmative case, $\mathcal{Z}_{\mathbf{c}}(F) = \{\mathbf{c}\}$ and we can use Theorem 3.6 to extend $\mathbf{c}$ to a unique formal power series solution

$$z = -\frac{1}{8} - \frac{1}{2}x.$$

It is straightforward to verify that z is a singular solution of $F(y) = 0$.

Similarly, let $\tilde{\mathbf{c}} = (-\frac{1}{8}, -\frac{1}{2}, 1, c_3)$, where c_3 is an arbitrary constant in $\mathbb{K}$. Using item 2 of Theorem 5.6, we find $\tilde{\mathbf{c}}$ can be extended into a formal power solution of $F(y) = 0$ if and only $c_3 = 0$. In the affirmative case, $\mathcal{Z}_{\tilde{\mathbf{c}}}(F) = \{\tilde{\mathbf{c}}\}$ and we find that

$$\tilde{z} = -\frac{1}{8} - \frac{1}{2}x + \frac{1}{2}x^2$$

is the corresponding solution.

Actually, one can verify that $z, \tilde{z}$ are all the formal power series solutions of $F(y) = 0$ with $[x^0]S_F(y) = 0$. Therefore, the set of formal power series solutions of $F(y) = 0$ at the origin is equal to

$$\{z, \tilde{z}\} \cup \mathcal{S},$$

where

$$\mathcal{S} = \{y \in \mathbb{K}[[x]] \mid F(y) = 0 \text{ and } [x^0]S_F(y) \neq 0\},$$

which can be determined by Proposition 2.3.

Below is an example of non-vanishing AODE of order 2.

Example 5.9. Consider the following second order AODE:

$$F = x(y'' - 1)^2 + (y - x)(y' - 1) = 0.$$

By computation, we find that F is a second non-vanishing differential polynomial.

Let $\mathbf{c}_1 = (100/9, 1, -1/9, 0, -1/120, 0, c_6) \in \mathcal{Z}(\mathcal{J}_4)$, where c_6 is an arbitrary constant in $\mathbb{K}$. Furthermore, the differential polynomial F is first non-vanishing at $\tilde{\mathbf{c}}_1 = \pi_4(\mathbf{c}_1)$. We find that $\mathbf{g}_{F,\tilde{\mathbf{c}}_1} = (\frac{20(2-t)}{9})$ and $\mathbf{q}_{F,\tilde{\mathbf{c}}_1} = 2$. From item 2 of Theorem 5.6, we see that $\mathbf{c}_1$ can be extended into a formal power solution if and only it can be extend to a zero point of $\mathcal{J}_5(F)$. However, a direct calculation shows that this is not possible for any $c_6 \in \mathbb{K}$. Therefore, $\mathcal{Z}_{\mathbf{c}_1}(F) = \emptyset$.

Let $\mathbf{c}_2 = (0, 0, 1 - i, \frac{3(1+i)}{4}, \frac{-3+4i}{8}, \frac{-2-9i}{64}, c_6) \in \mathcal{Z}(\mathcal{J}_4)$, where c_6 is an arbitrary constant in $\mathbb{K}$. Furthermore, the differential polynomial F is first non-vanishing at $\tilde{\mathbf{c}}_2 = \pi_4(\mathbf{c}_2)$. We find that $\mathbf{g}_{F,\tilde{\mathbf{c}}_2} = -2(3 + t)i$ and $\mathbf{q}_{F,\tilde{\mathbf{c}}_2} = 0$. From item 2 of Theorem 5.6, the initial tuple $\mathbf{c}_2$ can be extended into a formal power solution if and only it can be extended to a zero point of $\mathcal{J}_5(F)$. We find that this is only possible for $c_6 = \frac{3(47-11i)}{160}$. In this case, $\mathcal{Z}_{\mathbf{c}_2}(F) = \{\mathbf{c}_2\}$ and $\mathbf{c}_2$ can be extended uniquely to the formal power series solution

$$z \equiv \frac{1-i}{2}x^2 + \frac{1+i}{8}x^3 - \frac{3-4i}{192}x^4 - \frac{2+9i}{120 \cdot 64}x^5 + \frac{47-11i}{240 \cdot 160}x^6 \mod x^7.$$

6 Statistical study of AODEs

In this section, we do some statistical study of AODEs from the collection of differential equations in [Kam13]. The source codes and a demo notebook are freely available at the following website:

https://yzhang1616.github.io/fps/fps.html

In summary, there are 836 irreducible AODEs in Kamke's collection. Some of them has parameters (such as a, b, c) in the polynomial coefficients. In the generic case, there are at least non-vanishing 749 ($\approx 89.59\%$) AODEs. Below is a detailed figure [4].

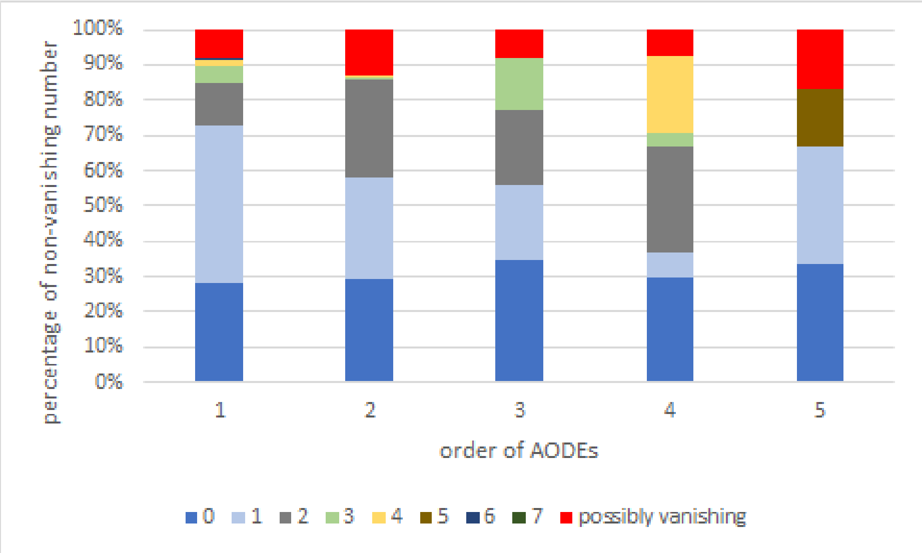

Figure 1: An overview of irreducible AODEs in Kamke's collection.

From experiments, we observe that there are examples with parameters, where the non-vanishing property for the generic case is different from that of the particular case. Below is a concrete example.

Example 6.1. Consider the following AODE with a parameter $a \in \mathbb{K}$:

$$F = (y')^2 + y^3 + ax = 0.$$

By computation, we see that $F = 0$ is a first non-vanishing AODE if $a \neq 0$. Nevertheless, if $a = 0$, then we conclude from Example 5.4 that F is vanishing.

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