ON SEQUENCES ASSOCIATED TO INVARIANT THEORY

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Abstract. We study the enumerative and analytic properties of sequences constructed using tensor invariant theory. The octant sequences are constructed from G_2 and the quadrant sequences from $SL(3)$. In each case we show that those sequences are related by binomial transforms. The first three octant sequences and the first four quadrant sequences are listed in the On-Line Encyclopedia of Integer Sequences (OEIS). These sequences all have interpretations as enumerating two dimensional lattice walks but for the octant sequences the boundary conditions are unconventional. These sequences are all P-recursive and we give the corresponding recurrence relations. In all cases the differential operators are of third order and have the remarkable property that they can be solved to give closed formulae for the ordinary generating functions in terms of classical Gaussian hypergeometric functions. Moreover, we show that the octant sequences and the quadrant sequences are related by the branching rules for the inclusion of $SL(3)$ in G_2.

Contents

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1. Introduction

We study two families of sequences each constructed using tensor invariant theory. The theory is summarised in § 2. The first family is the octant sequences. The first octant sequence is constructed from the seven dimensional fundamental representation of the exceptional simple algebraic group G_2. This sequence is studied initially in [24] and subsequently in [30]. We will refer to this sequence as T_3. The second family is the quadrant sequences. The first quadrant sequence is constructed from the direct sum of the three dimensional vector representation of $SL(3)$ and its dual. The sequences are defined as the dimension of the subspace of invariant tensors in the tensor powers of the representation. In both cases we extend to a family of sequences by adding copies of the trivial representation to the initial representation. Combinatorially, this means that each family consists of the iterated binomial transforms of the first sequence.

The first three sequences of the octant sequences have entries in OEIS and these are shown in Figure 1. The sequence A059710 is the first octant sequence T_3.

a (OEIS tag)	0	1	2	3	4	5	6	7	8	9
A059710	1	0	1	1	4	10	35	120	455	1792
A108307	1	1	2	5	15	51	191	772	3320	15032
A108304	1	2	5	15	52	202	859	3930	19095	97566

Figure 1. The first family of sequences **a**, their OEIS tags and their first terms.

The sequence T_3 has not appeared in the combinatorics literature and in § 3 we give the following combinatorial interpretations of this sequence:

Theorem 1.1. *The sequence T_3 enumerates*

- *hesitating tableaux of height 2, empty shape, and no singleton*
- *set partitions with no singleton and no enhanced 3-crossing*
- *sequences $(x_1, x_2, \ldots, x_n)$ such that $1 \leqslant x_i < i$ with no weakly decreasing subsequence of length 3*

These all follow from known combinatorial interpretations of the binomial transform of T_3. These are corollaries to Theorem 3.1.

The starting point for Theorem 1.1 is the interpretation of the sequences in the first family as lattice walks in the plane restricted to the region $0 \leqslant y \leqslant x$ together with an unconventional boundary condition on the line $x = y$. This interpretation is an application of the theory of Kashiwara crystals. Let V be a finite dimensional representation of a reductive algebraic group. Consider the sequence of dimensions of the subspace of invariant tensors in the tensor powers of V. The theory of Kashiwara crystals gives an interpretation of this sequence in terms of walks in the weight lattice restricted to the dominant chamber with boundary conditions which depend on the representation. Let E_3 be the second octant sequence A108307. Using this interpretation, we have the following theorem.

Theorem 1.2. *The sequence E_3 is the binomial transform of the sequence T_3, i.e., for $n \geq 0$,*

$$E_3(n) = \sum_{i=0}^{n} \binom{n}{i} T_3(i).$$

This gives an unexpected connection between the invariant theory of G_2 and the combinatorics of set partitions.

In § 3.3 we study the ordinary generating functions of these sequences. First we derive the recurrence relation of T_3 in Theorem 1.3.

Theorem 1.3. *The sequence T_3 is determined by the initial conditions $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and the recurrence relation that for $n \geq 0$,*

$$\begin{aligned} (1) \quad & 14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) \\ & + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0. \end{aligned}$$

The recurrence relation in Theorem 1.3 was conjectured by Mihailovs and we give three independent proofs in § 3.3. The corresponding differential equation can be solved to give a closed formula for the generating function in terms of the hypergeometric function ${}_2F_1$. The differential operators and recurrence relations for the other two sequences in Figure 1 are derived in [7] and the closed formulae are given in OEIS.

The first four sequences of the quadrant sequences have entries in OEIS and these are shown in Figure 2. The most well-known of these sequences is the third sequence which enumerates Baxter permutations.

a (OEIS tag)	0	1	2	3	4	5	6	7	8
A151366	1	0	2	2	12	30	130	462	1946
A236408	1	1	3	9	33	131	561	2535	11971
A001181	1	2	6	22	92	422	2074	10754	58202
A216947	1	3	11	47	225	1173	6529	38265	233795

FIGURE 2. The second family of sequences **a**, their OEIS tags and their first terms.

These sequences also have interpretations as lattice walks. These walks are restricted to the quadrant $0 \leqslant x, y$ and there are no boundary conditions as both three dimensional fundamental representations of $SL(3)$ are minuscule. The first sequence is studied in [6] where it is defined in terms of lattice walks. A differential operator and recurrence relation are given. The fourth sequence is introduced in [29]. It is defined as the number of noncrossing 2-coloured set partitions; the lattice path interpretation is given and a differential operator and recurrence relation are found. The identification of the third sequence with the sequence enumerating Baxter permutations can be proved by several methods. One method (Theorem 4.12) is to show that they both satisfy the same recurrence relation and initial conditions. Another method is to prove that they both enumerate axis walks in the octant, using [15], [11], [9] or our derivation utilizing branching rules. The second sequence does not have a published reference. We identify this sequence with the OEIS entry by giving the combinatorial interpretation and showing that the binomial transform is the third sequence in § 4.

The ordinary generating functions for these sequences are studied in § 4.4. We give a recurrence relation which includes k as a parameter. A closed formula for the generating function of the Baxter sequence is given in OEIS without a reference. We give a different, but equivalent, closed formula.

These two families of sequences are related since $SL(3)$ is a maximal subgroup of G_2. Furthermore, the restriction of the seven dimensional fundamental representation is the direct sum of the two three dimensional representations and one copy of the trivial representation. More generally, the restriction of the representation for the k-th sequence in octant family is the $(k+1)$-st sequence in the quadrant family. This means these two sequences are related by the branching rules for the inclusion of $SL(3)$ in G_2. This gives a conceptual explanation of the result that axis walks in the octant family correspond the quadrant family. This proof is not bijective. Bijective proofs in particular cases are known. Once this result is known for one value of k it follows for all values of k (not bijectively).

2. Invariant theory

In this section we describe the way the representation theory of algebraic groups gives rise to sequences. Let G be a reductive complex algebraic group. Let V_λ be the irreducible representation of G with highest weight λ.

Definition 2.1. *Let V be a (finite dimensional) representation of G and λ a dominant weight. The* sequence associated to *(G, V, λ), denoted $\mathbf{a}_{V,\lambda}$, is the sequence whose n-th term is the multiplicity of V_λ in the tensor power $\otimes^n V$.*

Our main interest is in the case $\lambda = 0$. In this case V_λ is the trivial representation and the n-th term of the sequence $\mathbf{a}_{V,\lambda}$ is the dimension of the subspace of invariant tensors in $\otimes^n V$. The sequence $\mathbf{a}_{V,0}$ is denoted by $\mathbf{a}_V$.

Example 2.2. *The simplest example is to take $G = SL(2)$ and V to be the two dimensional defining representation. The odd terms of the sequence $\mathbf{a}_V$ are zero and the even terms are the Catalan numbers.*

The generating function for the sequence of Catalan numbers is algebraic. This is a special case of a more general result. Take $G = SL(2)$ and V to be any representation. Then the generating function of the sequence $\mathbf{a}_V$ is algebraic.

Next we discuss the case V is a fundamental representation of a classical simple Lie algebra of rank two. These sequences have all been studied and are known to be holonomic.

Example 2.3. *Take $G = SL(3)$ and V to be the three dimensional defining representation. In the sequence $\mathbf{a}_V$ every third term in non-zero and the other terms are zero. Taking every third term gives A005789. The n-th term in this sequence is the number of standard tableaux of rectangular shape (n, n, n).*

Example 2.4. *Take $G = Sp(4)$ of type C_2 and V to be the four dimensional defining representation. Under the isomorphism $Sp(4) \cong Spin(5)$, V is the spin representation. The odd terms in the sequence $\mathbf{a}_V$ are zero and the even terms are sequence A005700. This sequence is the number of 3-noncrossing perfect matchings on $2n$ points and also the number of nested pairs of Dyck paths of length $2n$.*

Example 2.5. *Take $G = Sp(5)$ of type B_2 and V to be the five dimensional defining representation. This gives the sequence A095922. This is the binomial transform of the sequence whose odd terms are zero and whose even terms are the squares of Catalan numbers.*

The theory of Kashiwara crystals gives combinatorial interpretations of these sequences. There is a crystal associated to each representation. The crystal associated to the irreducible representation V_λ is denoted by C_λ. These are the connected crystals. Crystals have a tensor product with the property that the crystal associated to the tensor product of representations is the tensor product of the crystals associated to the representations. The trivial crystal is C_0 and has one element. Moreover, the crystal associated to the direct sum of representations is the disjoint union of the crystals associated to the representations.

Let C be a crystal. The elements of $\otimes^n C$ are words of length n in the elements of C. The analogue of the subspace of invariant tensors is the subset of invariant words in $\otimes^n C$.

The crystal version of Definition 2.1 is:

Definition 2.6. *Let C be a (finite) crystal of G and λ a dominant weight. The* sequence associated to *(G, C, λ), denoted $\mathbf{a}_{C,\lambda}$, is the sequence whose n-th term is the multiplicity of the crystal C_λ in the tensor power $\otimes^n C$.*

Our main interest is in the case $\lambda = 0$. In this case C_λ is the trivial crystal and the n-th term of the sequence $\mathbf{a}_{C,\lambda}$ is the cardinality of the set of invariant words of length n. The sequence $\mathbf{a}_{C,0}$ is denoted by $\mathbf{a}_C$.

The key result connecting these is:

Proposition 2.7. *If C is the crystal of the representation V then $\mathbf{a}_{C,\lambda} = \mathbf{a}_{V,\lambda}$ for all dominant weights λ.*

2.1. **Binomial transform.** In this subsection we give basic properties of the binomial transform. Identities associated to the binomial transform are given in [8].

The *binomial transform operator* is denoted by $\mathcal{B}$. This is a linear operator acting on sequences. Given the sequence $\mathbf{a}$ with n-th term $a(n)$, the *binomial transform of* $\mathbf{a}$ is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} a(i).$$

Binomial transforms arise naturally for sequences associated to the representations of reductive complex algebraic groups.

Lemma 2.8. *Assume that $\mathbf{a}_{V,\lambda}$ is the sequence associated to (G, V, λ) as specified in Definition 2.1. Then $\mathbf{a}_{V\oplus\mathbb{C},\lambda} = \mathcal{B}\mathbf{a}_{V,\lambda}$.*

The binomial transform also arises naturally for lattice walks restricted to a domain.

Lemma 2.9. *Assume that a sequence $\mathbf{a}$ enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ is given by adding a new step corresponding to the zero element of the lattice; that is, S is replaced by the disjoint union $S\coprod\{0\}$ without changing the domain.*

The following is a corollary to both Lemma 2.8 and Lemma 2.9

Corollary 2.10. *Assume that $\mathbf{a}_{C,\lambda}$ is the sequence associated to (G, C, λ) as specified in Definition 2.6. Let $*$ be the trivial crystal. Then $\mathbf{a}_{C\coprod *} = \mathcal{B}\mathbf{a}_{C,\lambda}$.*

We can also consider iterations of the binomial transform.

Lemma 2.11. *Given the sequence $\mathbf{a}$ with n-th term $a(n)$. For $k \in \mathbb{Z}$, the k-th binomial transform of $\mathbf{a}$ is given by*

$$(\mathcal{B}^k\mathbf{a})(n) = \sum_{i=0}^{n} k^{n-i}\binom{n}{i} a(i) \quad \text{for each} \quad n \in \mathbb{N}.$$

The binomial transform can also be regarded as an operator on the generating function of a sequence. Let $G(t) = \sum_{n=0}^{\infty} a(n)t^n$ be the generating function of the sequence $\mathbf{a}$. We denote the generating function of $\mathcal{B}^k\mathbf{a}$ by $\mathcal{B}^kG$. Then we have

Lemma 2.12. *For $k \in \mathbb{Z}$, the k-th binomial transform of $G(t)$ is*

$$(\mathcal{B}^k G)(t) = \frac{1}{1-k\,t} G\left(\frac{t}{1-k\,t}\right).$$

3. Octant sequences

In this section we consider a sequence associated to the representation theory of the exceptional simple algebraic group G_2 by the construction in Definition 2.1. Then we relate this sequence and its binomial transforms to known sequences of hesitating tableaux of height 2 and vacillating tableaux of height 2. These were introduced in [13].

The algebraic group G_2 has dimension 14 and rank 2. It can be constructed as the automorphism group of the (complex) octonions. By construction, the octonions are then an 8 dimensional representation of G_2. The unit spans a one dimensional invariant subspace and the imaginary octonions are an invariant complementary subspace. This 7 dimensional representation of G_2 is a fundamental representation.

3.1. **Root systems.**

The root system of G_2 is shown in Figure 3 with the fundamental chamber shaded. The two simple roots are α and β with α short and β long. The two fundamental weights are λ and θ. The highest weight representation with highest weight λ is the seven dimensional representation we are interested in. The highest weight representation with highest weight θ is the adjoint representation, so θ is the highest root.

These are related by

$$\lambda = 2\,\alpha + \beta, \qquad \theta = 3\,\alpha + 2\,\beta.$$

The element ρ is defined in terms of the roots as half the sum of the positive roots. The six positive roots are

$$\alpha, \beta, \alpha+\beta, 2\,\alpha+\beta, 3\,\alpha+\beta, 3\,\alpha+2\,\beta.$$

This gives $\rho = 5\,\alpha + 3\,\beta$.

The element ρ is also defined in terms of the weights as the sum of the fundamental weights. This gives $\rho = \lambda + \theta$.

3.2. **Lattice walks.**

In this subsection, we give a combinatorial proof of Theorem 1.2. Let G be G_2, V be the seven dimensional fundamental representation, and C be the associated crystal. We now apply the general theory of § 2 to this case.

The highest weight words in tensor powers of C correspond to lattice walks in the weight lattice of G_2 restricted to the dominant chamber. These lattice walks are studied in [24] and [30]. Since V has dimension 7, C has cardinality 7, and the seven steps are shown in Figure 4.

The highest weight words correspond with these steps, restricted to the dominant chamber and with the extra condition that the step $(0,0)$ is not permitted at a weight (k,k) for $k \geqslant 0$.

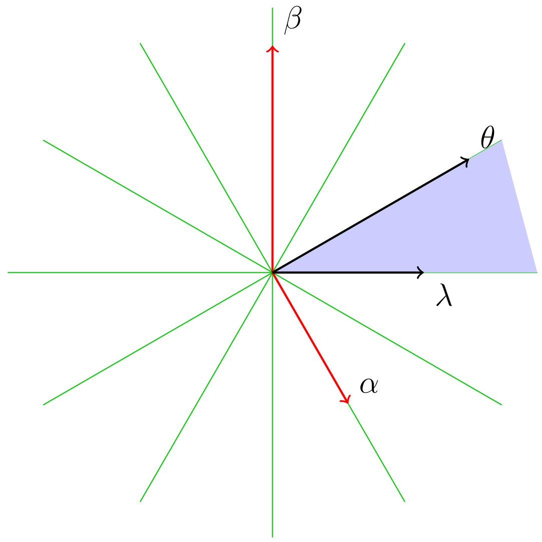

FIGURE 3. G_2 root system

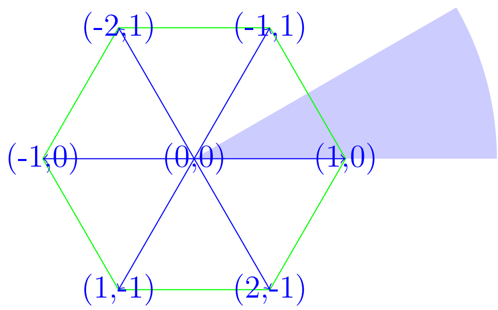

FIGURE 4. Steps in weight lattice

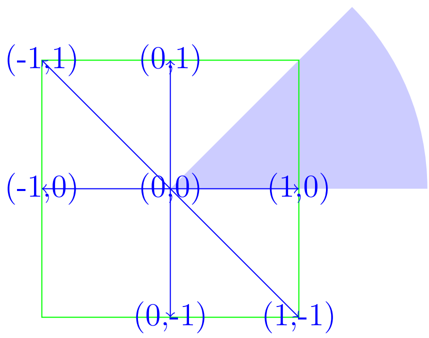

FIGURE 5. Steps in octant

A *hesitating tableau* of semilength n is a walk in the Young lattice with n steps. Each step is one of the following pairs of moves on the Young lattice:

- do nothing, add a cell

- remove a cell, do nothing
- add a cell, remove a cell

The *shape* is the final partition.

A hesitating tableau of *height* h is a hesitating tableau such that every partition in the sequence has height at most h.

There is a lattice walk interpretation of hesitating tableaux of height 2 given in [7].

A hesitating tableau of height 2 can be interpreted as a walk in $\mathbb{Z}^2$ by identifying partitions with at most two nonzero rows with the set

$$\{(x, y) \in \mathbb{Z}^2 | x \geqslant y \geqslant 0\}$$

There are 8 steps which are shown in Figure 5.

There are eight steps since there are:

- two ways to do nothing then add a cell,
- two ways to remove a cell then do nothing
- four ways to remove a cell then add a cell

Two of the four ways to remove a cell then add a cell give the step $(0, 0)$, namely add a cell on the first row then remove this cell and add a cell on the second row then remove this cell. It is always allowed to add and then remove a cell on the first row. The step which adds and removes a cell on the second row is not permitted on the line $x = y$.

Proof of Theorem 1.2. We compare the two descriptions of the walks. The steps of the walks for the sequence T_3 are shown in Figure 4 and the steps of the walks for the sequence E_3 are shown in Figure 5.

In order to compare these, we first make the change of coordinates $(x, y) \to (x, y \pm x)$. This identifies the six non-zero steps, as well as the two domains.

By Lemma 2.9, it remains to compare the zero steps. There is one zero step in the T_3 case and two in the E_3 case. In the E_3 case we have the zero step which adds and then removes a cell on the second row. The boundary condition is that this step is not allowed on the line $x = y$. After the change of coordinates, this is the same boundary condition as the zero step in the T_3 case. The second zero step in the E_3 case adds and then removes a cell in the first row. This is always allowed.

$\square$

Theorem 3.1. *For each $n \geqslant 0$ and $r, s \geqslant 0$ there is a correspondence between highest weight words of weight (r, s) and length n in the crystal $C \coprod *$ and hesitating tableaux of height 2, semilength n and shape $(r + s, s)$.*

Proof. The proof is the observation that the two descriptions of the lattice walks agree under a simple change of coordinates. The point (r, s)

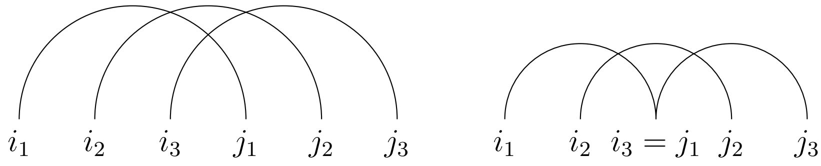

FIGURE 6. A 3-crossing and an enhanced 3-crossing

in the weight lattice is dominant if $r, s \geqslant 0$. This point corresponds to the two part partition $(r + s, s)$.

It is straightforward to check that under this change of coordinates, the eight steps correspond, the two domains correspond, and the extra condition on a step of weight $(0, 0)$ correspond. $\square$

There is a variation of Theorem 3.1 for vacillating tableaux. A *vacillating tableau* of semilength n is an excursion in the Young lattice with n steps. Each step is one of the following pairs of moves on the Young lattice:

- do nothing twice
- do nothing, add a cell
- remove a cell, do nothing
- remove a cell, add a cell

A vacillating tableau of *height* h is a vacillating tableau such that every partition in the sequence has height at most h.

There is a lattice walk interpretation of vacillating tableaux of height 2 also given in [7]. There are nine steps since there are six steps which change the shape and three steps which leave the shape unaltered.

Vacillating tableaux were introduced in [13] in the context of set partitions. A *set partition* will mean a partition of the ordered set

$$[n] = \{1, 2, \dots, n\}$$

into non-empty pairwise disjoint subsets. The *standard representation* of a set partition is a graph with vertex set $[n]$. The edges are arcs connecting pairs of elements in a block which are adjacent in the numerical order on the block.

A *singleton* in a set partition is a block with one element.

A *k-crossing* in a set partition is a k-subset of arcs, $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$i_1 < i_2 \cdots < i_k < j_1 < j_2 \cdots < j_k$$

An *enhanced k-crossing* in a set partition is a k-subset of arcs, $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$i_1 < i_2 \cdots < i_k \leqslant j_1 < j_2 \cdots < j_k$$

Theorem 3.2. *For each $n \geqslant 0$ and $r, s \geqslant 0$ there is a correspondence between highest weight words of weight (r, s) and length n in the crystal*

$C \coprod * \coprod *$ and vacillating tableaux of height 2, semilength n and shape $(r+s, s)$.

Proof. The proof is essentially the same as the proof of Theorem 3.1. □

The main innovation in [13] is the construction of a bijection between set partitions of $[n]$ and vacillating tableaux of semilength n and empty shape. The set partition has a k-crossing if and only if some partition in the vacillating tableau has length at least k. In particular, for $k = 3$, this is a bijection between 3-noncrossing set partitions and vacillating tableaux of height 2 and empty shape.

There is a correspondence between hesitating tableaux of semilength n, empty shape and height h and set partitions of n with no enhanced $h+1$-crossing.

This is proved in [27] and [29, Proposition 5.8]. In [20] the sequence A108307 is called $NC_{0,3}$, the sequence A108304 is called $NC_{1,3}$; and the binomial transform relation is the case $k = 3$ in Theorem 1.

A corollary is that there is a correspondence between invariant words of length n in C and set partitions of $[n]$ with no singleton and no enhanced 3-crossing.

There is another combinatorial interpretation based on [31] and [27].

Definition 3.3. *An* inversion sequence *of length n is a sequence $(x_1, x_2, \ldots, x_n)$ such that $1 \leqslant x_i \leqslant i$.*

A *singleton* in the inversion sequence $(x_1, x_2, \ldots, x_n)$ is an $i \in [n]$ such that $x_i = i$.

Proposition 3.4. *For $n \geqslant 0$ there is a bijection between set partitions of $[n]$ with no enhanced 3-crossing and no singletons and inversion sequences of length n with no weakly decreasing subsequence of length 3 and no singletons.*

Proof. There is a bijection in [31] between set partitions of $[n]$ with no enhanced 3-crossing and inversion sequences of length n with no weakly decreasing subsequence of length 3. This bijection preserves singletons. □

3.3. **Algebra.** In this subsection, we give three proofs of Theorem 1.3, with different flavors. The first two proofs utilize Theorem 1.2 and a result of Bousquet-Mélou and Xin [7] on partitions that avoid 3-crossings. The last one is a direct proof which relies on the connection with G_2 walks.

The first proof is based on Theorem 1.2, on Proposition 2 in [7] and on the method of creative telescoping [33, 22] for the summation of (bivariate) holonomic sequences.

Proposition 3.5. *[7, Proposition 2] The number $E_3(n)$ of partitions of $[n]$ having no enhanced 3-crossing is given by $E_3(0) = E_3(1) = 1$, and for $n \geq 0$,*

$$8(n+3)(n+1)E_3(n) + \left(7n^2 + 53n + 88\right)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0. \tag{2}$$

Equivalently, the associated generating function $\mathcal{E}(t) = \sum_{n\geq 0} E_3(n)t^n$ satisfies

$$t^2(1+t)(1-8t)\frac{d^2}{dt^2}\mathcal{E}(t) + 2t\left(6 - 23t - 20t^2\right)\frac{d}{dt}\mathcal{E}(t) + 6\left(5 - 7t - 4t^2\right)\mathcal{E}(t) - 30 = 0.$$

First Proof of Theorem 1.3. By Theorem 1.2, we have

$$T_3(n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_3(k).$$

Set $f(n,k) = (-1)^{n-k}\binom{n}{k}E_3(k)$. By Proposition 3.5, and by the closure properties for holonomic functions, it follows that $f(n,k)$ is holonomic. Thus, we can apply Chyzak's algorithm [22] for creative telescoping to derive a recurrence relation for T_3. In particular, using Koutschan's Mathematica package `HolonomicFunctions.m` [23] that implements Chyzak's algorithm, we find exactly the recurrence equation in Theorem 1.3.

□

The detailed calculation involved in the above proof can be found in [4].

The second proof is also based on Proposition 3.5 and on Theorem 1.2, namely on the relation between the generating functions of $T_3(n)$ and of $E_3(n)$ implied by Theorem 1.2.

Second Proof of Theorem 1.3. Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$. By Theorem 1.2 and Lemma 2.12,

$$\mathcal{T}(t) = \frac{1}{1+t}\cdot\mathcal{E}\left(\frac{t}{1+t}\right).$$

We know from Proposition 3.5 a differential equation for $\mathcal{E}(t)$. By (univariate) closure properties of D-finite functions, we deduce a differential equation for $\mathcal{T}(t)$, and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence in Theorem 1.3. □

Let G be a reductive complex algebraic group and V be a representation of G. If G is connected then the sequence $\mathbf{a}_V$ associated to (G,V) can sometimes be written as an algebraic residue. The general principle is discussed in [19] and in detail in [21].

Let K be the character of V and let Δ be given by

$$\Delta = \sum_{w \in W} \varepsilon(w)\,[w(\rho) - \rho]$$

Here W is the Weyl group, $\varepsilon\colon W \to \{\pm 1\}$ is the sign character and ρ is half the sum of the positive roots. These are both elements of the group ring of the root lattice of G. If we choose a basis of the root lattice then these become Laurent polynomials where the number of indeterminates is the rank of G.

For the representations that appear in this paper, the n-th term of the sequence $\mathbf{a}_V$ is the constant term in the Laurent polynomial $\Delta\,K^n$ and the generating function is the algebraic residue of the rational function $\Delta/(xy - txyK)$.

Example 3.6. *In Example 2.2, the Laurent polynomials are*

$$\Delta = (1 - x^{-2}), \qquad K = (x - x^{-1}).$$

Then the constant term of the Laurent polynomial

$$(1 - x^{-2})(x - x^{-1})^{2n}$$

is

$$\binom{2n}{n} - \binom{2n}{n+2},$$

which is a well known expression for the n-th Catalan number. This can be proved directly using the reflection principle.

It is a general result that an algebraic residue is D-finite, see [28]. This shows that the generating functions of these sequences are D-finite and hence P-recursive.

The third proof of Theorem 1.3 relies on G_2 walks and the method in [5].

The following definition is given in [25] in the paragraph following Conjecture 3.3.

Definition 3.7. *The n-th term $T_3(n)$ is the constant term of the Laurent polynomial $\Delta\,K^n$, where*

$$K = (1 + x + y + x\,y + x^{-1} + y^{-1} + (xy)^{-1}),$$

and Δ is the Laurent polynomial

$$\begin{aligned}\Delta = x^{-2}y^{-3}(&x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2}\\ &+ x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).\end{aligned}$$

Third Proof of Theorem 1.3. Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. By Definition 3.7, $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $\Delta/(1 - tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $\Delta/(xy - txyK)$, and thus it is D-finite [28] and is annihilated by a linear differential operator that cancels the contour integral of $\Delta/(xy - txyK)$ over a cycle. Using the integration algorithm for multivariate rational functions in [3] we compute the following operator of order 6, that cancels $\mathcal{T}(t)$:

$$\begin{aligned} L_6 = {} & t^5\,(t+1)\,(7\,t-1)\,(2\,t+1)^2\,\partial^6 + \\ & 3\,t^4\,(2\,t+1)\left(168\,t^3+211\,t^2+40\,t-11\right)\partial^5 + \\ & 6\,t^3\left(2100\,t^4+3475\,t^3+1616\,t^2+79\,t-61\right)\partial^4 + \\ & 6\,t^2\left(11200\,t^4+17400\,t^3+7556\,t^2+268\,t-273\right)\partial^3 + \\ & 36\,t\left(4200\,t^4+6100\,t^3+2442\,t^2+54\,t-77\right)\partial^2 + \\ & 36\,\left(3360\,t^4+4540\,t^3+1646\,t^2+16\,t-35\right)\partial + \\ & 20160\,t^3+25200\,t^2+8064\,t, \end{aligned}$$

where $\partial = \frac{\partial}{\partial t}$ denotes the derivation operator with respect to t.

The operator L_6 factors as $L_6 = QL_3$, where

$$Q = (2\,t+1)\,t^3\partial^3 + (24\,t+13)\,t^2\partial^2 + 6\,(12\,t+7)\,t\partial + 48\,t + 30,$$

and

$$\begin{aligned} L_3 = {} & t^2\,(2\,t+1)\,(7\,t-1)\,(t+1)\,\partial^3 + 2\,t\,(t+1)\left(63\,t^2+22\,t-7\right)\partial^2 + \\ & \left(252\,t^3+338\,t^2+36\,t-42\right)\partial + 28\,t\,(3\,t+4)\,. \end{aligned} \tag{3}$$

This shows that $f(t) := L_3(\mathcal{T}(t))$ is a solution of the differential operator Q. Hence, by denoting $f(t) = \sum_{n\geq 0} f_n t^n$, one deduces that for all $n \geq 0$ we have

$$2\,(n+2)\,f_n + (n+6)\,f_{n+1} = 0.$$

One the other hand, from $\mathcal{T}(t) = 1 + t^2 + O(t^3)$, it follows that $f_0 = 0$, therefore $f(t) = 0$, in other words $\mathcal{T}(t)$ is also a solution of L_3. From there, deducing the recurrence relation of Theorem 1.3 is immediate. □

3.4. **Closed formulae.** In this subsection, we give two closed formulae for the generating function $\mathcal{T}(t)$ of sequence A059710 in terms of classical Gaussian hypergeometric functions ${}_2F_1$.

The operator L_3 in (3) factors itself as $L_3 = L_2 L_1$, where

$$L_2 = t^2\,(2\,t+1)\,(7\,t-1)\,(t+1)\,\partial^2 + \frac{t(t+1)P_1}{P}\partial + \frac{P_2}{P^2},$$

and

$$L_1 = \partial - \frac{\frac{\mathrm{d}}{dt}(P/t^5)}{P/t^5}$$

with

$$P_1 = 3136\,t^6 + 7560\,t^5 + 5744\,t^4 + 1592\,t^3 - 90\,t^2 - 131\,t - 9,$$

$$\begin{aligned} P_2 = {} & 131712\,t^{11} + 719712\,t^{10} + 1626800\,t^9 + 2014088\,t^8 + 1498264\,t^7 + \\ & 665620\,t^6 + 146508\,t^5 - 4560\,t^4 - 11138\,t^3 - 2663\,t^2 - 244\,t - 7 \end{aligned}$$

and

$$P = 28\,t^4 + 66\,t^3 + 46\,t^2 + 15\,t + 1.$$

Clearly, the operator L_1 has a basis of solutions formed of $\{P/t^5\}$. Let $\{f_1, f_2\}$ be a basis of solutions of L_2. Then one can show that the solution $\mathcal{T}(t)$ of L_3 is

$$\frac{P}{t^5} \cdot \int (C_1 f_1 + C_2 f_2) \frac{t^5}{P},$$

for some constants C_1 and C_2 to be determined.

Using algorithms for solving second order differential equations, as described in [5], one deduces that f_1 and f_2 have hypergeometric expressions. After identifying the constants C_1 and C_2, we derive the following explicit formula for $\mathcal{T}(t)$. Define the polynomials:

$$S = (7\,t - 1)\,(t + 1)\,(2\,t + 1)^2,$$
$$U = (11\,t - 1)\,(46\,t^3 - 78\,t^2 + 15\,t - 1),$$

and

$$V = 11870\,t^7 - 6934\,t^6 - 13371\,t^5 + 1115\,t^4 + 1112\,t^3 - 300\,t^2 + 29\,t - 1,$$

as well as the rational function

$$\phi = \frac{27\,(t+1)\,t^2}{(1-t)^3}.$$

Then the generating function $\mathcal{T}(t)$ is equal to

$$\frac{P}{30\,t^5} \int \frac{S}{P^2\,(t-1)^2} \left[U\,{}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ & 1\end{matrix}; \phi\right) + \frac{V}{(1-t)^3}\,{}_2F_1\left(\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix}; \phi\right)\right] dt.$$

It appears that this expression can be further simplified by introducing the additional rational functions

$$R_1 = \frac{(t+1)^2\,(214\,t^3 + 45\,t^2 + 60\,t + 5)}{t - 1}$$

and

$$R_2 = 6\,\frac{t^2\,(t+1)^2\,(101\,t^2 + 74\,t + 5)}{(t-1)^2}.$$

Then the simpler expression for $\mathcal{T}(t)$ is

$$\mathcal{T}(t) = \frac{1}{30\,t^5}\left[R_1 \cdot {}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ & 2\end{matrix}; \phi\right) + R_2 \cdot {}_2F_1\left(\begin{matrix}\frac{2}{3} & \frac{4}{3}\\ & 3\end{matrix}; \phi\right) + 5\,P\right].$$

Following the approach in [5, §3.3], one can obtain an alternative expression for $\mathcal{T}(t)$ by using a more geometric flavor. The key point is that the denominator of $W/(xy - txyK)$ is a family of *elliptic* curves, thus integrating $W/(xy - txyK)$ over a small torus amounts to computing the periods of the two forms (of the first and second kind). Working out the details, this approach yields an expression for $\mathcal{T}(t)$ in terms of the Weierstrass invariant

$$g_2 = (t-1)\left(25\,t^3+21\,t^2+3\,t-1\right)$$

and of the j-invariant

$$J = \frac{(t-1)^3\left(25\,t^3+21\,t^2+3\,t-1\right)^3}{t^6\,(1-7\,t)\,(2\,t+1)^2\,(t+1)^3}$$

of our family of curves. As in [5], we introduce the expression

$$H(t) = \frac{1}{{g_2}^{1/4}}\cdot {}_2F_1\left(\begin{matrix}\frac{1}{12} & \frac{5}{12}\\ & 1\end{matrix};\frac{1728}{J}\right).$$

Then the generating function $\mathcal{T}(t)$ is equal to

$$\frac{P}{6\,t^5}+\frac{(7\,t-1)\,(2\,t+1)\,(t+1)}{360\,t^5}\Big(\left(155\,t^2+182\,t+59\right)(11\,t+1)\,H\,(t)+ \left(341\,t^3+507\,t^2+231\,t+1\right)(5\,t+1)\,H'(t)\Big).$$

Moreover, one can use the methods in [5] to deduce that the generating function $\mathcal{T}(t)$ is a transcendental power series and that the asymptotic behavior of its n-th coefficient $T_3(n)$ is

$$T_3(n) \sim C\cdot\frac{7^n}{n^7},\quad \text{where } C = \frac{4117715}{864}\frac{\sqrt{3}}{\pi} \approx 2627.6.$$

4. Quadrant sequences

In this section we consider a family of sequences associated to the representation theory of the simple algebraic group $SL(3)$ by the construction Definition 2.1. The algebraic group $SL(3)$ has dimension 8 and rank 2. Let V be the three dimensional vector representation and V^* the dual representation. These are the two fundamental representations of $SL(3)$.

Definition 4.1. *For $k \geqslant 0$, the quadrant sequences $\mathcal{S}_k$ are the sequence associated to $(SL(3), V\oplus V^*\oplus k\,\mathbb{C})$.*

We will show (Corollary 4.13) that the first four sequences S_0, S_1, S_2, S_3 are identical to the sequences given in Figure 2. By Lemma 2.8, those sequences are also related by binomial transforms.

By the general theory in Proposition 2.7 these sequences enumerate walks in the weight lattice of $SL(3)$. Equivalently, these are walks in the positive quadrant which is our reason for calling them *quadrant sequences*.

The n-th term of the sequence $\mathcal{S}_0$ enumerates A_2-webs with n boundary points, see [24]. If we use the representation theory of $GL(3)$ (instead of $SL(3)$) then we can translate this to a combinatorial interpretation in terms of rectangular semistandard tableaux with three rows.

Let $\mathrm{SST}(\lambda;\alpha)$ be the set of semistandard tableaux of shape λ and weight α. Here λ is a partition, α is a composition and $|\lambda| = |\alpha|$.

Proposition 4.2. *The first three quadrant sequences have the following interpretations in terms of rectangular semistandard tableaux.*

(1) *The n-th term of the sequence $\mathcal{S}_0$ is the number of rectangular semistandard tableaux with three rows whose weight, α, is a composition of length n such that $\alpha_i \in \{1,2\}$ for $1 \leqslant i \leqslant n$.*
(2) *The n-th term of the sequence $\mathcal{S}_1$ is the number of rectangular semistandard tableaux with three rows whose weight, α, is a composition of length n such that $\alpha_i \in \{0,1,2\}$ for $1 \leqslant i \leqslant n$.*
(3) *The n-th term of the sequence $\mathcal{S}_1$ is the number of rectangular semistandard tableaux with three rows whose weight, α, is a composition of length n such that $\alpha_i \in \{1,2,3\}$ for $1 \leqslant i \leqslant n$.*
(4) *The n-th term of the sequence $\mathcal{S}_2$ is the number of rectangular semistandard tableaux with three rows whose weight, α, is a composition of length n such that $\alpha_i \in \{0,1,2,3\}$ for $1 \leqslant i \leqslant n$.*

Proof. There are easy bijections with the respective walks. A semistandard tableaux can be regarded as a sequence of partitions where the i-th partition is the shape given by taking the cells whose entry is at most i. For a semistandard tableau with (at most) three rows this gives a sequence of vectors $(\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0$. Each such vector maps to the point $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$. This constructs a map from semistandard tableaux with three rows to walks in the two dimensional lattice which stay in the non-negative quadrant. This map gives all bijections. $\square$

Proposition 4.3. *The first three quadrant sequences have the following expressions.*

(1) *The n-th term of the sequence $\mathcal{S}_0$ is*
$$\sum_{m\geqslant 0} \sum_{\substack{a,b,c\geqslant 0\\ a+b=n\\ a+2b=3m}} \binom{n}{a,b} \left|\mathrm{SST}(3^m;1^a2^b)\right|$$
(2) *The n-th term of the sequence $\mathcal{S}_1$ is*
$$\sum_{m\geqslant 0} \sum_{\substack{a,b,c\geqslant 0\\ a+b+c=n\\ b+2c=3m}} \binom{n}{a,b,c} \left|\mathrm{SST}(3^m;0^a1^b2^c)\right|$$
(3) *The n-th term of the sequence $\mathcal{S}_1$ is*
$$\sum_{m\geqslant 0} \sum_{\substack{a,b,c\geqslant 0\\ a+b+c=n\\ a+2b+3c=3m}} \binom{n}{a,b,c} \left|\mathrm{SST}(3^m;1^a2^b3^c)\right|$$

(4) *The n-th term of the sequence $\mathcal{S}_2$ is*

$$\sum_{m\geqslant 0} \sum_{\substack{a,b,c,d\geqslant 0\\ a+b+c=n\\ b+2c+3d=3m}} \binom{n}{a,b,c,d} \left|\mathrm{SST}(3^m;1^b2^c3^d)\right|$$

Proof. These follow directly from Proposition 4.2 together with the result that $|\mathrm{SST}(\lambda;\alpha)| = |\mathrm{SST}(\lambda;\alpha')|$ if α' is a reordering of α. □

Note that it is clear from these expressions that $\mathcal{S}_1$ is the binomial transform of $\mathcal{S}_0$ and that $\mathcal{S}_2$ is the binomial transform of $\mathcal{S}_1$. This uses the observation that $\left|\mathrm{SST}(3^m;1^a2^b)\right| = \left|\mathrm{SST}(3^{m+c};1^a2^b3^c)\right|$ for all $a,b,c,m \geqslant 0$ such that $a+2b=m$.

The four sequences $\mathcal{S}_k$ for $k=0,1,2,3$ appear in OEIS.

- The sequence $\mathcal{S}_0$ appears as sequence A151366. This sequence is defined in [6] using lattice walks.
- The sequence $\mathcal{S}_1$ appears as sequence A236408. There is no published reference for this sequence and we discuss this sequence in § 4.2 below.
- The sequence $\mathcal{S}_2$ appears as sequence A001181. This is the sequence of Baxter permutations. These two sequences can be seen to be the same by noting that Proposition 4.3 together with the known formula for the number of rectangular partitions agrees with the formula in [14] for the number of Baxter permutations. Alternatively one can check that both sequences satisfy the same recurrence relation and initial conditions. A bijective proof is given by noting that that there is an easy bijection using the combinatorial interpretation in Proposition 4.2 and [16, Theorem 2]. Alternative proofs are given in [10, Proposition 23], and [15, Theorem 3].
- The sequence $\mathcal{S}_3$ appears as sequence A216947. This sequence is defined in [29] using 2-colored noncrossing set partitions. This paper also gives an easy bijection with the lattice paths.

The Definition 2.1 generalises to give sequences $\mathbf{a}_{V\oplus V^*\oplus k\,\mathbb{C},\lambda}$. Here $\lambda=(r,s)$ for $r,s\geqslant 0$ and the sequence enumerates lattice walks which start at $(0,0)$ and end at (r,s).

4.1. **Root sytems.** The root system of $SL(3)$ is shown in Figure 7 with the fundamental chamber shaded. The two simple roots are α and α'. The two fundamental weights are λ and λ'. The fundamental representations are the three dimensional vector representation and its dual.

The element ρ is

$$\rho = \alpha + \alpha' = \lambda + \lambda'.$$

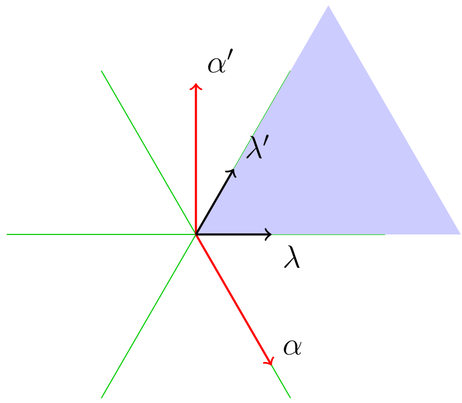

FIGURE 7. A_2 root system

All four entries in the OEIS give a recurrence relation for each quadrant sequence. These recurrence relations all agree with, or are consequences of, the recurrence relations in § 4.4. If the recurrence relation in OEIS has a proof then we can check that the initial terms agree and so we have an alternative proof that the two sequences agree. If the recurrence relation in OEIS is conjectural then our results prove that these recurrence relations are satisfied by the sequence.

As a consequence of the identification of $\mathcal{S}_2$ with the Baxter sequence we can give a formula for the terms of $\mathcal{S}_k$. This formula is obtained by substituting the formula in [14] for the terms in the Baxter sequence into the formula in Lemma 2.11 for iterated binomial transforms.

Other known results that are consequences are: A bijective proof of the following is given in [10, Proposition 23] and [15, Theorem 3] and a recent paper by Yan.

Proposition 4.4. *The set of hesitating tableaux of length n which end with a single row partition is equinumerous with the set of Baxter permutations on $[n]$.*

A bijective proof of the following is [15, Corollary 14].

Proposition 4.5. *The set of vacillating tableaux of length n which end with a single row partition is equinumerous with the set of 2-coloured noncrossing set partitions of $[n]$.*

4.2. **Increasing maps.** In this subsection we give a combinatorial interpretation of the sequence $\mathcal{S}_1$. This depends on the closely related combinatorial interpretations of the Baxter sequence in [1] (mosaic floor plans), [2] (plane bipolar orientations), and [26] (diagonal rectangulations). All three of these interpretations are equivalent to oriented st-maps.

A *map* is a connected graph embedded in the plane with no edge-crossings,considered up to isotopy. The vertices and edges of the map are those of the graph. The faces of the map are the connected components of the complement of the graph in the plane. The *outer face* is unbounded, the inner faces are bounded.

An *oriented map* is a map together with an orientation of each edge. An oriented map is an *st-map* if it has a unique source, s, and a unique sink, t, both on the outer face.

An *increasing map* is an oriented map together with an increasing labelling of the vertices. If the map has r vertices then the labelling, ℓ, is a bijection between the vertices and the set $\{1, 2, \ldots, r\}$. A labelling is increasing if whenever there is a directed path from vertex v to vertex w then $\ell(v) < \ell(w)$. Let O be an increasing st-map. Then $\ell(s) = 1$ and $\ell(t) = r$.

A *simple increasing st-map* is a increasing st-map whose underlying graph has no multiple edges.

Proposition 4.6. *Let $a(n)$ be the number of increasing st-maps with n edges and let $a_S(n)$ be the number of simple increasing st-maps with n edges. Then*

$$a(n+1) = \sum_{k=0}^{n} \binom{n}{k} a_S(k+1)$$

Proof. There is a bijection between simple increasing st-maps with each edge labelled by a positive integer and increasing st-maps since we can replace each edge labelled by m with m copies of the same edge. This is a bijection between increasing st-maps with n edges and pairs consisting of a simple increasing st-map with k edges and a labelling of the edges by positive integers whose sum is n. The result follows from the observation that the number of sequences of positive integers of length k with sum n is $\binom{n+1}{k+1}$. □

A Corollary is a proof that the conjectured recurrence relation for A236408 is correct.

Plane bipolar orientations arise in the unpublished notes by James Cranch, Pasting Diagrams, as the morphisms in a strict monoidal category.

First we construct the category. The set of objects is $\mathbb{N}$ so we have an object $[n]$ for each $n \in \mathbb{N}$. Let O be an oriented st-map. One of the oriented paths going from s to t has the outer face on its right: this path is the *right border* of O, and its length is the *right outer degree* of O. The *left border* and *left outer degree* is defined similarly. An oriented st-map, O, is a morphism $[l] \to [r]$ where l is the left outer degree of O and r is the right outer degree. Composition of O and O' is only defined if the right outer degree of O is the left outer degree of O'. In this case, the composite $O \circ O'$ is the oriented st-map given by identifying the right border of O with the left border of O'. The identity morphism of $[n]$ is a path with n edges.

This category has a tensor product. On objects this is just $[n] \otimes [m] = [n+m]$ so the monoid of objects is $\mathbb{N}$. The tensor product $O \otimes O'$ is always defined and is the plane bipolar orientation given by identifying the sink of O with the source of O'. It is straightforward to check that this tensor product is strictly associative and that these are compatible in the sense that they satisfy the interchange condition $(O \circ O') \otimes (P \circ P') = (O \otimes P) \circ (O' \otimes P')$. Hence this is a strict monoidal category, $\mathcal{P}$. Simple plane bipolar orientations are the morphisms of a strict monoidal subcategory, $\mathcal{P}_S$.

The monoidal category $\mathcal{P}$ is the free strict monoidal category whose monoid of objects is $\mathbb{N}$ with a morphism $\alpha(n,m)\colon [n] \to [m]$ for $n, m > 0$. The monoidal subcategory $\mathcal{P}_S$ is the free strict monoidal category whose monoid of objects is $\mathbb{N}$ with a morphism $\alpha(n,m)\colon [n] \to [m]$ for $n, m > 0$ with $\alpha(1,1)$ omitted. The morphism $\alpha(n,m)$ consists of a single inner face with n edges on the left border and m edges on the right border.

4.3. **Branching rules.** In this subsection we use the branching rules for the inclusion of $SL(3)$ in G_2 to relate the two families of sequences.

The interpretation using invariant theory gives a connection between the octant sequences and the quadrant sequences. This uses the fact that $SL(3)$ is a subgroup of G_2, in fact, a maximal subgroup. Then the restriction of the representation $V \oplus k\mathbb{C}$ from G_2 to $SL(3)$ is the representation $V \oplus V^* \oplus (k+1)\mathbb{C}$. This implies that each quadrant sequence $\mathbf{a}_{V \oplus k\,\mathbb{C},\lambda}$ can be written as a linear combination of the octant sequences $\mathbf{a}_{V \oplus k\,\mathbb{C},\mu}$ with coefficients which are independent of k. These coefficients are the branching rules for the inclusion of $SL(3)$ in G_2. A special case of this is the result that walks in the quadrant which end at $(0,0)$ correspond to walks in the octant which end on the x-axis, see [10], [15], [11], [9], [32].

The branching rules for the inclusion of the maximal subgroup

$$SL(3) \to G_2$$

are given in [18, (2.3)]. Let $V(r,s)$ be a highest weight representation of G_2 with highest weight (r,s) and let $U(p,q)$ be a highest weight representation of $SL(3)$ with highest weight (p,q) . Denote the multiplicity of $U(p,q)$ in the restriction of $V(r,s)$ to $SL(3)$ by $m^{(r,s)}_{(p,q)}$.

Proposition 4.7. *The generating function*

$$\sum_{r,s,p,q \geqslant 0} m^{(r,s)}_{(p,q)} x^p\, y^q\, X^r\, Y^s$$

is the rational function

$$\frac{(1-X)^{-1} - xyY(1-xyY)^{-1}}{(1-x\,X)(1-y\,X)(1-x\,Y)(1-y\,Y)} \tag{4}$$

Theorem 4.8. *Let V be a representation of G_2 and $V\downarrow$ the restriction to $SL(3)$. Then the number of axis-walks of length n for V is the number of excursions for $V\downarrow$.*

Proof. Putting $x=0$, $y=0$ in the generating function (4) gives

$$\sum_{r,s,p,q\geqslant 0} m^{(r,s)}_{(0,0)} X^r Y^s.$$

Putting $x=0$, $y=0$ in the rational function gives $(1-X)^{-1}$. This shows that

$$m^{(r,s)}_{(0,0)} = \begin{cases} 1 & \text{if } s=0, \\ 0 & \text{otherwise.} \end{cases}$$

∎

Example 4.9. *First we extend each octant sequence to a sequence of functions on the dominant weights of G_2, or, equivalently, on the octant $y\geqslant 0$, $x\geqslant y$. It follows from Lemma 2.11 that these can be written as polynomials in k.*

Then the octant sequences are given by

$$\begin{array}{|c} \mathit{1} \\ \hline \end{array} \quad \begin{array}{|cc} k & \mathit{1} \\ \hline \end{array} \quad \begin{array}{|ccc} & \mathit{1} & \\ 1+k^2 & 1+2\,k & \mathit{1} \\ \hline \end{array}$$

$$\begin{array}{|cccc} & 2+2\,k & \mathit{2} & \\ 1+3\,k+k^3 & 4+3\,k+3\,k^2 & 3+3\,k & \mathit{1} \\ \hline \end{array}$$

Then summing the entries on the bottom row give the sequence

$$1, 1+k, 3+2\,k+k^2, 9+9\,k+3\,k^2+k^3.$$

This proves Proposition 4.4 and Proposition 4.5.

4.4. **Recurrence equations.** In this subsection we give recurrence relations for the sequences $\mathcal{S}_k$. A recurrence relation for $\mathcal{S}_3$ is given in [29, § 4].

Lemma 4.10. *Let C_k be the generating function of $\mathcal{S}_k$, where $k\geq 0$. Then C_k is the constant coefficient $[x^0y^0]$ of $W/(1-tK)$, where*

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x} \tag{5}$$

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2y^2 + y^3 - \frac{y^2}{x}. \tag{6}$$

Let $C_2(n)$ be the n-th term of the sequence A216947 in the second family. Marberg [29, Theorem 1.7] showed that $C_2(n)$ is the constant term $[x^0y^0]$ of $W\tilde{K}^n$, where $\tilde{K}=K|_{k=3}$, the Laurent polynomials K and W are specified in (5) and (6). By Lemma 4.10, the sequence $\mathcal{S}_3$ is identical to the sequence A216947. Therefore, we have:

Proposition 4.11. *[29, Theorem 1.7] The n-th term $C_2(n)$ of the sequence $\mathcal{S}_3$ is given by $C_2(0) = 1, C_2(1) = 3$ and for $n \geq 0$:*

$$(n+5)(n+6) \cdot C_2(n+2) - 2(5n^2 + 36n + 61) \cdot C_2(n+1) \\ + 9(n+1)(n+4) \cdot C_2(n) = 0,$$

Equivalently, the associated generating function $\mathcal{C}(t) = \sum_{n \geq 0} C_2(n)t^n$ satisfies

$$72\mathcal{C}(t) + 4(-61 + 117t)\frac{d}{dt}\mathcal{C}(t) + 2(15 - 184t + 234t^2)\frac{d^2}{dt^2}\mathcal{C}(t) \\ + 2t(-6+7t)(-1+9t)\frac{d^3}{dt^3}\mathcal{C}(t) + (-1+t)t^2(-1+9t)\frac{d^4}{dt^4}\mathcal{C}(t) = 0.$$

Next, for $k = 0, 1, 2, 3$, we prove a uniform recurrence equation for the sequence $\mathcal{S}_k$. It is given by a single formula with k as a parameter. Moreover, we show that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to the sequences in the second family.

Theorem 4.12. *For $k = 0, 1, 2, 3$, the n-th term $a(n)$ of the sequence $\mathcal{S}_{3-k}$ satisfies the following recurrence equation:*

$$\begin{aligned} &(k-9)(k-1)k^2(n+1)(n+2)a(n) + \\ &\quad 2k(n+2)(2k^2n - 15kn + 8k^2 + 9n - 56k + 36)a(n+1) + \\ &(6k^2n^2 + 54k^2n - 30kn^2 + 114k^2 + 9n^2 - 254kn + 81n - 510k + 162)a(n+2) \\ &\quad + 2(2kn^2 + 24kn - 5n^2 + 70k - 56n - 153)a(n+3) \\ &\quad + (n+7)(n+8)a(n+4) = 0 \end{aligned} \tag{7}$$

Proof. By Lemma 2.8, sequences $\mathcal{S}_2, \mathcal{S}_1, \mathcal{S}_0$ are the first, second, and third inverse binomial transforms of S_3, respectively. Thus, it follows from Lemma 2.12that the generating function of S_k is

$$\mathcal{A}(t) := \frac{1}{1 + (3-k)t} \cdot \mathcal{C}\left(\frac{t}{1 + (3-k)t}\right) \quad \text{for} \quad k = 0, 1, 2, 3,$$

where $\mathcal{C}(t)$ is the generating function of S_3. Regarding k as a parameter in the above expression, we deduce the differential equation for $\mathcal{A}(t)$ in the claim by using Proposition 4.11 and closure properties of D-finite functions. By converting the differential equation for $\mathcal{A}(t)$, we get the corresponding recurrence equation for the sequence $a(n)$, which is exactly the recurrence equation in the claim. $\square$

Corollary 4.13. *The sequences $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to the sequences in the second family specified in Figure 2. Moreover, sequences in the second family are related by binomial transforms.*

Proof. In (7), by setting k to $0, 1, 2, 3$, we find that the corresponding recurrence operators are left multiples [12, page 618] of those of A216947, A001181, A236408, and A151366 specified in OEIS. It implies that

$\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ satisfy the same recurrence equations as the sequences in the second family. To verify they are the same sequences, we just need to check finitely many initial terms. The details of the verification can be found in [4]. Since $\mathcal{S}_k$'s are related by binomial transforms, so are sequences in the second family. $\square$

Closed formulae for these sequences can be obtained by the same methods as in Section 3.4.

Define the function

$$H(x) = {}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}; \frac{27x^2}{(1-2x)^3}\right).$$

Then the ordinary generating function of $\mathcal{S}_2$ is equal to

$$\frac{(x+1)^2\,(1-8x)}{3x^2\,(1-2x)^2}\left(H + \frac{12(1+20x-8x^2)(1-2x)}{(x+1)}H'\right) - \frac{(3x^2-x+1)}{(3x^2)}$$

The above formula is equivalent to the following formula for A001181 in OEIS.

$$-1 + \frac{1}{3x^2}\left[(x-1) + (1-2x)_2F_1\left(\begin{matrix}\frac{-2}{3} & \frac{2}{3} \\ & 1\end{matrix}; \frac{27x^2}{(1-2x)^3}\right) - \frac{(8x^3-11x^2-x)}{(1-2x)^2}{}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 2\end{matrix}; \frac{27x^2}{(1-2x)^3}\right)\right].$$

A closed formula for the generating function of $\mathcal{S}_k$ follows by substitution in Lemma 2.12.

The asymptotics is stated in [14] and a detailed account is given in [17, Theorem 1].

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