①

Recall:

1. Inverse matrix: $AA^{-1} = I$ and $A^{-1}A = I$.

2. ~~A is~~ Pivot test: A is invertible if and only if it has n pivots.

3. If $Ax = 0$ has nonzero solution, the A has no inverse

$\Rightarrow$ Diagonally dominant matrices are invertible

4. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

5. Gauss-Jordan method: $A^{-1}[A \ I] = [I \ A^{-1}]$

§2.6 Elimination = Factorization $A = LU$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$E_{21}^{-1}(E_{21}A) = E_{21}^{-1}U$$

$$A = E_{21}^{-1}U$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{\text{lower triangular}} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \leftarrow \text{upper triangular}$$

Assume: no row exchanges are involved.

3×3 case.

$$(E_{32}E_{31}E_{21})A = U \Rightarrow A = \underbrace{(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})}_{=L}U$$

Explanation and Examples

② First point: E^{-1} is lower triangular

$$E=\begin{bmatrix}1&0\\-l&1\end{bmatrix} \Rightarrow E^{-1}=\begin{bmatrix}1&0\\l&1\end{bmatrix}$$

Second point: L is the product of E_{ij}^{-1}'s.

* ~~Thrd~~ Third point: Eeach multiplier l_{ij} goes into i,j possition of L. (easy to compute

Ex: $E_{21}=\begin{bmatrix}1&0&0\\-l_{21}&1&0\\0&0&1\end{bmatrix}$, $E_{31}=\begin{bmatrix}1&0&0\\0&1&0\\-l_{31}&0&1\end{bmatrix}$, $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-l_{32}&1\end{bmatrix}$

Then $L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}$

Ex 1.

$$A=\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix} \xrightarrow{r_2-\frac{1}{2}r_1} \begin{bmatrix}2&1&0\\0&\frac{3}{2}&1\\0&1&2\end{bmatrix} \xrightarrow{r_3-\frac{2}{3}r_2}$$

$$U=\begin{bmatrix}2&1&0\\0&\frac{3}{2}&1\\0&0&\frac{4}{3}\end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\0&\frac{2}{3}&1\end{bmatrix}$$

$A=LU$

Better balance from LDU

③

$A = LU$

$$U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots \\ & d_2 & u_{23} & \cdots \\ & & \ddots & \vdots \\ & & & d_n \end{bmatrix}$$, ~~d_i is~~ d_i's are pivots of A.

$$U = \underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & & 0 \\ & 0 & \ddots & \\ & & & d_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots \\ & 1 & u_{23}/d_2 & \cdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}}_{U}$$

(we use it in Gauss-Jordan method)

$A = LDU$

Ex: $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

One Square System = Two Triangular System.

Solve $Ax = b$:

1. Factor $A = LU$ by elimination
2. Solve $\underset{\text{elimination}}{Lc = b}$ and $\underset{\text{back subs}}{Ux = c}$

Correctness: $L(Ux) = Ax = b$

Ex 3.

$$\begin{cases} u + 2v = 5 \\ 4u + 9v = 21 \end{cases} \xrightarrow{\text{elimination}} \begin{cases} u + 2v = 5 \\ v = 1 \end{cases}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}$$

④

$Lc = b$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} c = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$Ux = c$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The Cost of Elimination.

~~Ass~~ Assume A is $n \times n$, consider $Ax = b$

1. Factor $A = LU$ requires about $\frac{1}{3}n^3$ multiplication and $\frac{1}{3}n^3$ substractions.

2. Solve $Lc = b$ and then $Ux = c$ needs n^2 multiplication and n^2 subtractions