(1) Recall: 1. 4 ways to multiply matrices

① dot product

② column

③ row

④ columns multiply rows

2. Block multiplication is allowed when block shapes match correctly

§ 2.5 Inverse Matrices

Let $A$ be a square Matrix

Def $A$ is invertible if there exists $A^{-1}$ such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

Not all matrices have inverses. Ex: cyclic difference matrix

Note 1: $A$ is invertible if and only if it has $n$ pivots.
(I will show you later)

Note 2: $A^{-1}$ is unique

Assume $BA = I$ and $AC = I$. Then $B = C$

Pf: $B(AC) = (BA)C \Leftrightarrow BI = IC \Leftrightarrow B = C$

Note 3: If $A$ is invertible, then $Ax = b \Rightarrow x = A^{-1}b$

$$Ax = b$$

$$(A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$$

$$\Leftrightarrow x = A^{-1}b$$

Note 4: If $A$ is invertible, then $Ax = 0 \Rightarrow x = 0$

~~Note~~ Ex 1: $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $Ax = 0$ has solution $x = (2, -1)$.

$A$ is not invertible

Note 6.

If $A = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ with $d_i \neq 0$, $i = 1, \cdots, n$.

Then $A^{-1} = \begin{bmatrix} d_1^{-1} & & 0 \\ & \ddots & \\ 0 & & d_n^{-1} \end{bmatrix}$.

The inverse of $AB$

If $A$ and $B$ are invertible, so is $A \cdot B$

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

Pf: 
$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Note: Inverse come in reverse order.

$$(ABC)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$$

Ex. 2. (Inverse of Elimination matrix)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex 3.

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}, \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix}$ is invertible by $E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

③ Computing $A^{-1}$ by Gauss-Jordan Elimination

Let $A$ be a (3 by 3) ~~square~~ matrix. Assume $A^{-1} = [x_1 \ x_2 \ x_3]$

Then $A \cdot A^{-1} = A[x_1 \ x_2 \ x_3] = I = [e_1 \ e_2 \ e_3]$

$$A x_1 = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A x_2 = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A x_3 = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Idea: the Gauss-Jordan method computes $A^{-1}$ by solving the above equations together

Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[A \ I] = [A \ e_1 \ e_2 \ e_3] = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 + \frac{1}{2} r_1} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 + \frac{2}{3} r_2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$U$

Jordan: reduce $U$ to $I$ by elimination.

$$\xrightarrow{r_2+\frac{3}{4}r_3}\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$\xrightarrow{r_1+\frac{2}{3}r_2}\begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$\xrightarrow[\text{its pivot}]{\text{divide row by}}\begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [\,I \quad A^{-1}\,]$$

Why?

Gauss-Jordan: $A^{-1}[\,A \quad I\,] = [\,I \quad A^{-1}\,]$

Singular versus Invertible — Assume $A$ is $n \times n$

Pivot test: $A^{-1}$ exists ~~exactly when~~ $\Longleftrightarrow$ $A$ has a full set of $n$ pivots.

"$\Leftarrow$": With $n$ pivots, $Ax_i = e_i$ has a solution, $i = 1, \cdots, n$.

Set $A^{-1} = [x_1, \cdots, x_n]$.

Ex 6. Let $L$ be a triangular matrix.

Then $L$ is invertible $\Longleftrightarrow$ no diagonal entries are zero

"$\Leftarrow$"

$$[\,L \quad I\,] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(pivot test)

$$\xrightarrow[r_3-4r_1]{r_2-3r_1}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix}$$

5

$$\xrightarrow{r_3 - 5r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -5 & 1 \end{array}\right]$$

$$\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -5 & 1 \end{array}}_{= L^{-1}}$$

Recognizing an Invertible Matrix

Theorem: Diagonally dominant matrices are invertible.

Let A be a ~~sq~~ square matrix with: ~~each $a_{ii}$ on the~~

On ~~ev~~ each row,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Ex:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Pf: Fact: A is invertible $\Leftrightarrow$ $Ax = 0$ has only zero vector solution.

Assume $x$ is a nonzero solution of $Ax = 0$

Let $|x_i|$ be the largest component of $x$.

Then

$$a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = 0$$

$$\Leftrightarrow a_{ii}x_i = -\sum_{j \neq i} a_{ij}x_j$$

$$\Rightarrow |a_{ii}x_i| = \left|\sum_{j \neq i} a_{ij}x_j\right| \leq \sum_{j \neq i} |a_{ij}||x_j| \cancel{< |a_{ii}||x_i|}$$

$$\leq \sum_{j \neq i} |a_{ij}||x_i|$$

$$< |a_{ii}||x_i|, \text{ a contradiction.}$$