① Recall:

1. Elimination matrix E_{21}

$E_{21} A = \tilde{A}$ substracts a multiple of row 1 of A from row 2 of A

2. Exchange matrix P_{ij}

$P_{ij} A = \bar{A}$ ~~ech~~ exchange row i and j of A

3. $AB = A[b_1\ b_2\ b_3] = [Ab_1\ Ab_2\ Ab_3]$

4. Augmented matrix $[A\ b]$

$E_{21}[A\ b] = [E_{21}A\ E_{21}b]$

§2.4 Rules for Matrix Operations

Matrix ~~m~~addition and scalar multiplication

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}, \quad 2A = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}$$

Matrix ~~Ma~~ multiplication (4 ways)

Rule 1: To multiply AB: If A has n columns, B must have n rows (Fundamental Law of Matrix Multiplication)

Associative law: $(AB)C = A(BC)$

1. ~~Assu~~ Dot Product Way

Assume A is m by n and B is n by p. Then AB is m by p.

~~The entry in row i~~ 1. $(AB)_{ij} = $ (row i of A) $\cdot$ (column j of B).

Ex 1.

$$\begin{array}{c} r_1 \\ \end{array}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{array}{c} \begin{array}{cc} c_1 & \end{array} \\ \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \end{array} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

$r_1 \cdot c_1 = 1 \times 2 + 1 \times 3 = 5$

Ex 2.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0\times1 & 0\times2 & 0\times3 \\ 1\times1 & 1\times2 & 1\times3 \\ 2\times1 & 2\times2 & 2\times3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

$((n\times1)(1\times n) = (n\times n))$

dot product $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0\times1 + 1\times2 + 2\times3 = 8$

2. Column way

$$AB = A[b_1 \cdots b_p] = [Ab_1 \cdots Ab_p]$$

3. Row way

$$AB = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} B = \begin{bmatrix} r_1 B \\ \vdots \\ r_m B \end{bmatrix}$$

$r_1 = [r_{11} \cdots r_{1n}]$, $B = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}$

$r_1 \cdot B = r_{11} l_1 + \cdots + r_{1n} l_n$

4. Columns Multiply Rows

~~let~~ Assume A is $m\times \cancel{p}\, n$, ~~and~~ B is $\cancel{p}\, n\times p$,

~~$A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}, B =$~~

$A = [c_1\ c_2 \cdots c_n]$, $B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$.

$AB = c_1\cdot r_1 + c_2\cdot r_2 + \cdots + c_n\cdot r_n$ (a special case of block multiplication)

Ex. $A = \begin{array}{cc} c_1 & c_2 \\ \end{array}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} E & F \\ G & H \end{bmatrix}\begin{array}{c} r_1 \\ r_2 \end{array}$

$$A\cdot B = \begin{bmatrix} a \\ c \end{bmatrix}\cdot[E\ F] + \begin{bmatrix} b \\ d \end{bmatrix}[G\ H]$$

$$= \begin{bmatrix} aE+bG & aF+bH \\ cE+dG & cF+dH \end{bmatrix}$$

③ The Laws for Matrix Operations

1° addition laws are the same as number's.

2° multiplication laws: commutative law does not hold

Ex:
$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\neq$$
$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

other laws hold

laws for exponents: Let A be square matrix.

$$A^p = \underbrace{AA\cdots A}_{p \text{ times}}$$

$$(A^p)(A^q) = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

Block Matrices and Block Multiplication

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}$$

Ex: $Ax = b$

$[A \ b]$

$E[A \ b] = [EA \ \ Eb]$

Block multiplication If blocks of A can multiply that of B, then ~~B~~block multiplation is allowed

Ex 3. Let A be $m \times n$, B be $n \times p$

$$A = [\,c_1 \cdots c_n\,], \quad B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$A \cdot B = c_1 \cdot r_1 + c_2 \cdot r_2 + \cdots + c_n \cdot r_n$$ (~~B~~4th way for…)

④ $$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{array}{c}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\end{array}\begin{array}{c} r_1 \\ r_2 \end{array} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

~~Example~~ Ex 4. (Elimination by blocks

Let $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & * & * \\ 3 & * & * \\ 4 & * & * \end{bmatrix}$$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Set

$$E = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3 & 1 & 0 \\ -4 & 0 & 1 \end{array}\right], \quad A = \left[\begin{array}{c|cc} 1 & * & * \\ \hline 3 & * & * \\ 4 & * & * \end{array}\right]$$

$$EA = \left[\begin{array}{c|c} 1\times 1 & * \; * \\ \hline \begin{bmatrix} -3 \\ 4 \end{bmatrix}\times 1 + I\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{array}{cc} * & * \\ * & * \end{array} \end{array}\right]$$

$$= \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Block elimination

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \begin{bmatrix} A & B \\ 0 & \underbrace{D-CA^{-1}B}_{\text{Schur complement}} \end{bmatrix}$$

Review.

4 ways to multiply matrices,

1. dot product
2. column way
3. rows way
4. columns multiply rows.

⑤

~~[illegible]~~

2. Block multiplication is allowed when block shapes match correctly.