Recall:

1. $Ax = b \xrightarrow{\text{elimination}} Ux = c \xrightarrow{\text{back subs}} x = U^{-1}c$

2. pivot can not be zero

   multiplier $l_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$

3. If zero is in pivot position, exchange rows if there is a nonzero below it.

4. When breakdown happens, $Ax = b$ has no solution or many.
   $(0y = 8$ or $0y = 0)$

§ 2.3 Elimination Using Matrices

Matrices times Vectors and $Ax = b$

Let

$$A = \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \end{array} \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array}, \quad b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}, \quad \text{assume } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1° Column vector:

$$Ax = x_1 u + x_2 v + x_3 w$$

$$= x_1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

2° Row vector:

$$Ax = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 - 2x_3 \\ 4x_1 + 9x_2 - 3x_3 \\ -2x_1 - 3x_2 + 7x_3 \end{bmatrix}$$

The Matrix Form of One Elimination Step

$$Ax = b \xrightarrow[\text{②} - 2\text{①}]{\text{Elimination}} \tilde{A}x = \tilde{b}$$

$$b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \xrightarrow[\text{②} - 2\text{①}]{} \tilde{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

② Question: Is there a matrix E such that

$$Eb = \tilde{b}\ ?$$

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

$\overset{\shortparallel}{E_{21}}$ ← elimination ~~matrp~~ matrix (compare with identy matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

E_{21} subtracts a multiple of row 1 from row 2

Ex 2.

Identity $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Elimination $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$

Take ~~Ib~~ $b = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$, $l = 4$

$$\left(Ib = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}, \right)$$

$$E_{31}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 - 4\times 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$Ax = b \xrightarrow{E_{21}} E_{21}(Ax) = E_{21}b = \tilde{b}$$

$$\begin{array}{c} E_{21}(Ax) \\ \shortparallel \\ (E_{21}A)x \\ \shortparallel \\ \tilde{A} \end{array}$$

Matrix Multiplication

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & [illegible] & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

(3)

~~$E(Ax) = Eb$~~

1° $Ax = b \Rightarrow E(Ax) = Eb$

2° By matrix multiplication, $(EA)x = Eb$

$\Big\}$ $E(Ax) = (EA)x$

Let A, B, C be 3×3 matrices.

Associative law : $A(BC) = (AB)C$

$B = [b_1 \ b_2 \ b_3]$

Matrix multiplication $AB = A[b_1 \ b_2 \ b_3] = [Ab_1 \ Ab_2 \ Ab_3]$

Commutative law is false : often $AB \neq BA$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

~~Permutation matrix~~

Permutation matrix (Row exchange matrix)

Let

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$ (make pivot nonzero)

Is there a matrix P such that

$$PA = \tilde{A} \ ?$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{= P_{23}} \begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} \ & \tilde{v} & \ \end{matrix} \\ \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

$P_{23} \leftarrow$ Permutation matrix

Row exchange matrix P_{ij} is the identity matrix with rows i and j reversed.

④ The Augmented Matrix

concatenate [kan'kætəˌneit] v.

Let
$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

(Key idea: Elimination does same row operations to A and b)

Augmented matrix
$$[A \;\; b] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$E[A \;\; b] = [EA \;\; Eb]$$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

Review:

1. Elimination matrix E_{21}

   $E_{21} A = \tilde{A}$ subtracts a multiple of row 1 of A from row 2 of A

2. Exchange matrix P_{ij}

   $P_{ij} A = \bar{A}$ exchange row i and j of A

~~3. Augmented matrix~~

3. $AB = A[b_1 \; b_2 \; b_3] = [Ab_1 \; Ab_2 \; Ab_3]$

4. Augmented matrix $[A \;\; b]$

$$E_{21}[A \;\; b] = [E_{21}A \;\; E_{21}b].$$