① Recall:

1. Ax = combination of columns of A.

2. $Ax = b \Rightarrow x = A^{-1}b$ if A is invertible (columns are independent)

3. The cyclic matrix $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

has no inverse (colums are dependent). $Cx = 0$ has many solutions. $Cx = b$ may have no solutions

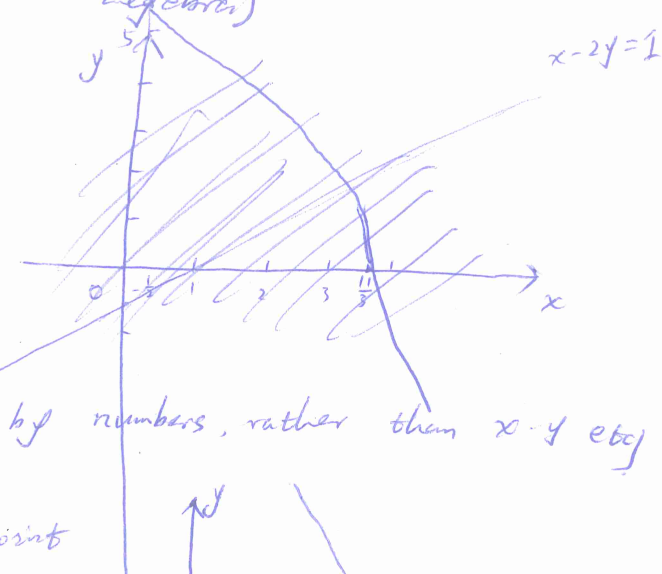
(illustrated by examples, not fully explained).

Chapter 2. Solving Linear Equations (fundamental problem of liner algebra)

§ 2.1 Vectors and Linear Equations.

~~Let~~

~~$A = [1 \; 2$~~

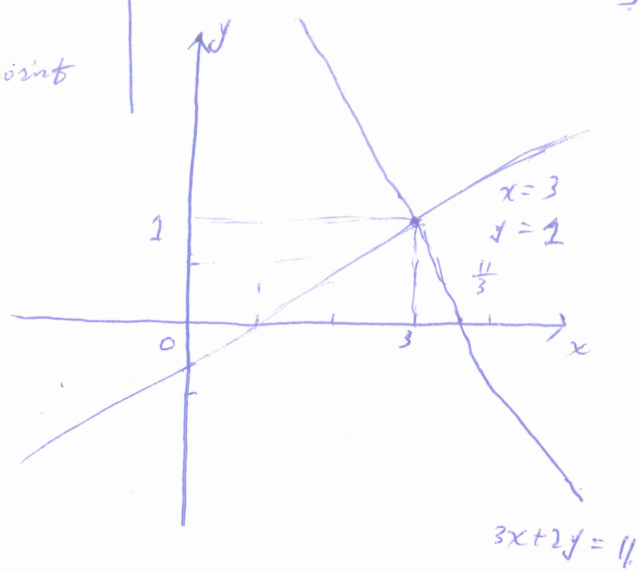


Consider linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \quad (1)$$

(linear equation: unknowns are multiplied by numbers, rather than $x \cdot y$ etc)

Row picture: two lines meets at a point $(3, 1)$



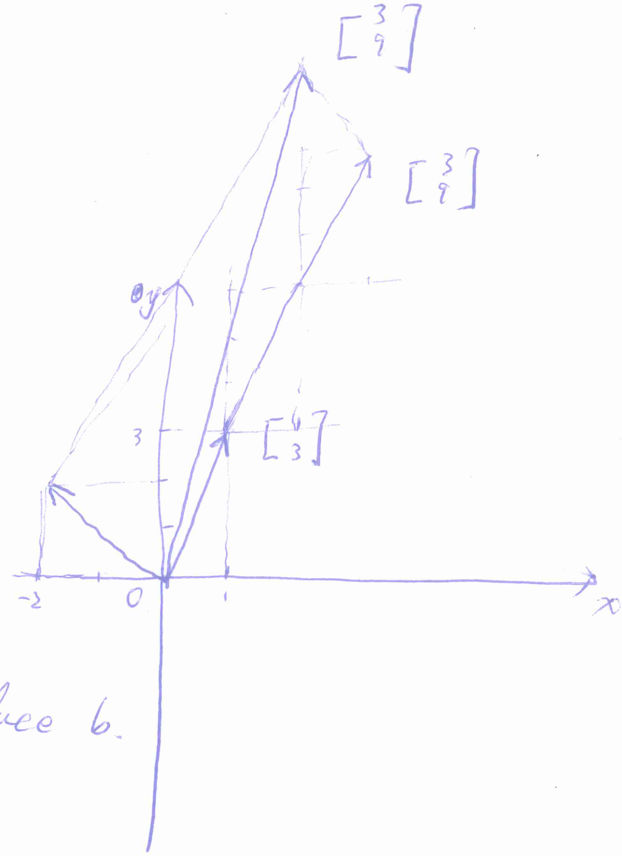
$$(1) \Leftrightarrow x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = b$$

② Take $x=3$, $y=1$,

$$3\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3\\9\end{bmatrix}$$

$$1\cdot\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

$$\begin{bmatrix}3\\9\end{bmatrix}+\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$



Column picture: take a linear combination of column vectors to ~~prod~~ produce b.

Let $A=\begin{bmatrix}1&-2\\3&2\end{bmatrix}$.

Then

(1) $\Longleftrightarrow$ $Ax=b$ matrix equation.

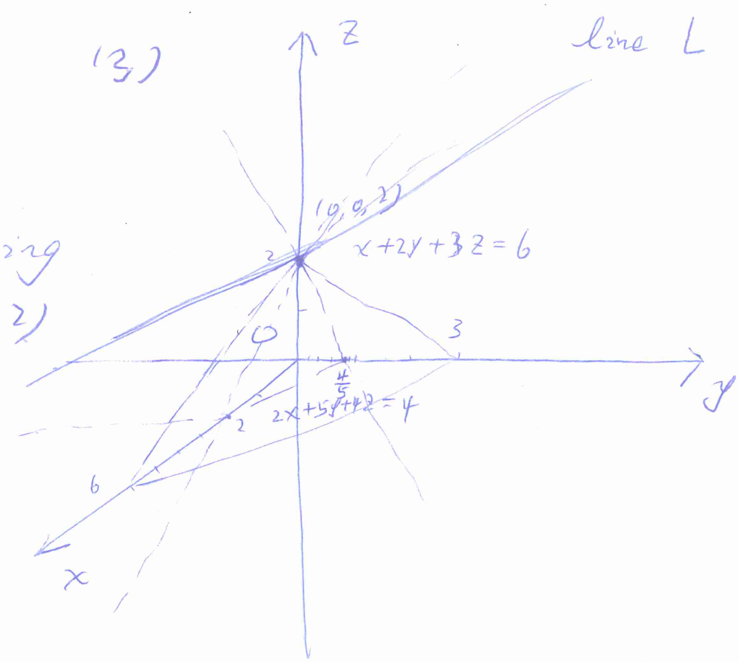
$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

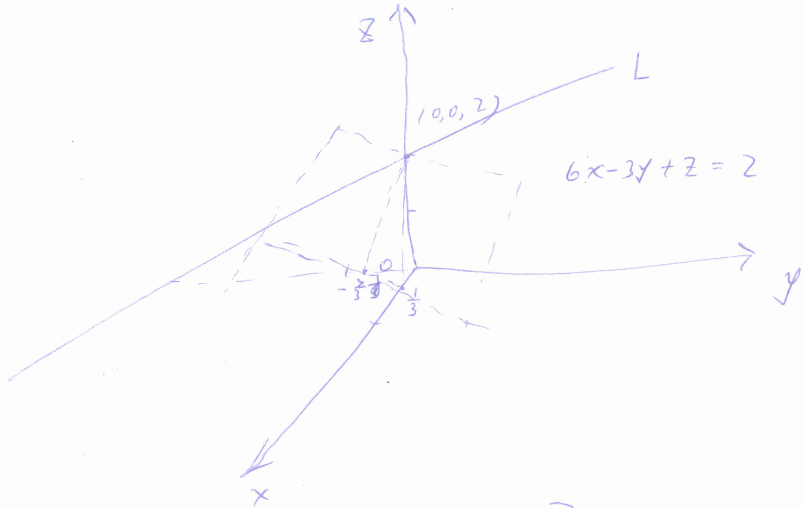
Dot products with rows
Combination of columns.

3-D case

$$\begin{cases} x+2y+3z=6\\ 2x+5y+2z=4\\ 6x-3y+z=2\end{cases} \quad (3)$$

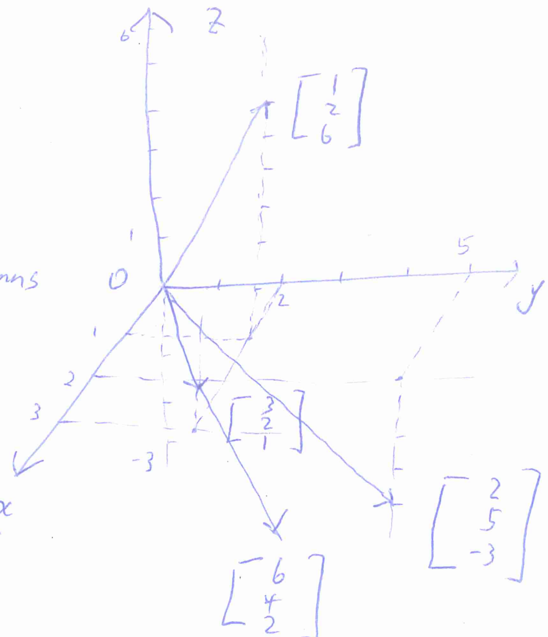
Row picture: three planes meeting at a point $(0,0,2)$





$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}=b$$

Take $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\2\end{bmatrix}$



Column picture: combination of columns to produce b.

(Remark: in the n-D case,

row picture: intersections of hyperplane

column picture: combination of columns to produce b).

The Matrix Form of Equations

Let

$$A=\begin{array}{c c} \begin{array}{ccc} u & v & w \end{array} & \\ \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}, \quad b=\begin{bmatrix}6\\4\\2\end{bmatrix}.$$

Matrix equation $Ax=b$

Multiplication by rows

$$Ax=\begin{bmatrix}r_1\cdot x\\r_2\cdot x\\r_3\cdot x\end{bmatrix}$$

4 Multiplication by columns

$$Ax = xu + yv + zw$$

Take $x = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, then

$$Ax = 2w = 2\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Ex. 1.

$$A = \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad x = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$Ax = \cancel{x}\ 4 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ identity matrix

$$I \cdot x = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x$$

Note: for each vector $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have

$I \cdot x = x$ (idendtity is similiar to 1 in the scalar case)

Matrix Notation

Let A be a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}$$

(5) Let A be a $m \times n$ matrix.

$$A = \begin{array}{c} \\ 1 \\ \vdots \\ \\ m \end{array} \begin{array}{c} 1 \; \cdots \; n \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Review: 1. Row picture for $Ax = b$

~~2.~~ 2-D case: two lines meet at a point

3-D case: ~~2~~3 planes meet at a point

2. Column picture for $Ax = b$

combination of columns of x gives b.

3. Multiplication by columns:

Ax: a combination of columns of A.