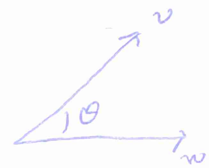
①

Recall:

1. $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$



$v \cdot w = v_1 \cdot w_1 + v_2 \cdot w_2$

2. $\|v\| = \sqrt{v \cdot v}$

$u = \dfrac{v}{\|v\|}$ is a unit vector

3. $v \cdot w = 0 \iff v$ and w are perpendicular.

4. $\cos\theta = \dfrac{v \cdot w}{\|v\| \cdot \|w\|}$

$\Rightarrow$ Schwarz inequality $|v \cdot w| \le \|v\| \cdot \|w\|$.

§1.3 Matrices.

Consider (three vectors in 3-D space)

$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Take $x_1 u + x_2 v + x_3 w$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

Set

matrix $A = \begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(matrix times vector via linear combination)

Define: $A \cdot x = \cancel{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}\ x_1 u + x_2 v + x_3 w = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$

~~1° (The matrix) A actsact acts on (the vector) x   of columns of A.~~

~~2° (The matrix) A times (the vector) x. (Ax is a linear combo~~

(Then b is a linear combination of columns of A.)

matrix times vector via dot product

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \end{array}$$

Set $r_1 = (1, 0, 0)$, $r_2 = (-1, 1, 0)$, $r_3 = (0, -1, 1)$

Then

$$A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$Ax = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

$$\begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

Thus,

$$Ax = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix}$$

Ax is also dot products with rows

Linear Equations.

Question: Given a matrix A and a vector b, compute a vector x such that

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Assume that

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then

$$Ax = b \iff \text{linear equations} \quad \begin{cases} x_1 = b_1 \\ -x_1 + x_2 = b_2 \\ -x_2 + x_3 = b_3 \end{cases} \quad (1)$$

$$\begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_1 + b_2 + b_3 \end{cases}$$

Ex.

$b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ gives $x = \begin{bmatrix} 1 \\ 1+3 \\ 1+3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$. (equation (1) is easy to solve because A is a triangular matrix.)

The Inverse Matrix

$\left(A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right)$

Consider $Ax = b \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

~~Let A~~ matrix A^{-1} is the inverse matrix of A

$Ax = b \Rightarrow x = A^{-1}b$ (analog of scalar case)

Note: not all matrices have inverse matrices.

Cyclic Differences.

Let

$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \; w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Consider the cyclic difference matrix C:

$$Cx = \begin{array}{c} \begin{array}{ccc} u & v & w^* \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

Given b, can we find x such that $Cx = b$.

(4) Take $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Then $\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

ex: $c = 3$.

Take $b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ Then $\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ (2)

~~Obersav~~ Observation: If $\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then

$$b_1 + b_2 + b_3 = 0$$

All linear combinations $x_1 u + x_2 v + x_3 w^*$ lie on the plane $b_1 + b_2 + b_3 = 0$

In (2), $1+3+5 = 9 \neq 0$. No solution for (2).

In this case, C is not invertible. ~~$Cx=b$~~

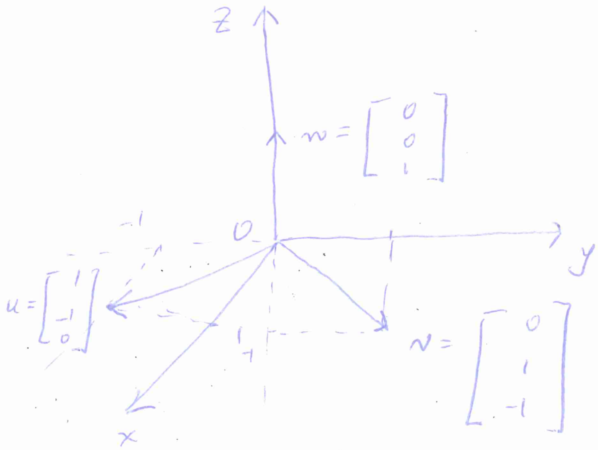
Independence and Dependence

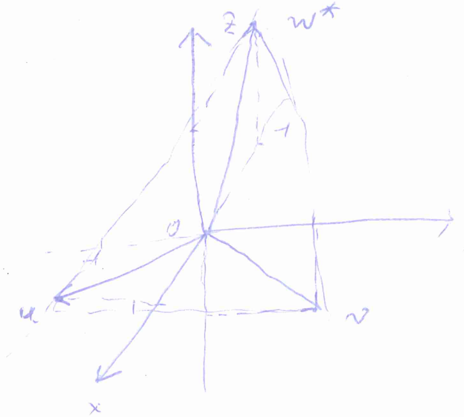
Let

$$A = \begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}, \quad \text{invertible} \quad Ax = b \Rightarrow x = A^{-1}b$$

$$C = \begin{array}{c} \begin{matrix} u & v & w^* \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}, \quad \text{not invertible} \quad Cx = b \not\Rightarrow x = C^{-1}b$$

C^{-1} does not exist.





Observation: w is not in the plane of u and v

w^* is in the plane of u and v

$u + v + w^* = 0$

$$w = cu + dv$$

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ -d \end{bmatrix} \iff \begin{cases} c = 0 \\ -c + d = 0 \\ -d = -1 \end{cases}$$ no solution!

A invertible: u, v, w are independent $\iff$ no combination except $0u + 0v + 0w = 0$ gives $b = 0$

B singular: u, v, w are dependent $\iff$ other combinations like $u + v + w^*$ give $b = 0$.

Review:

1. Matrix times vector: Ax = combinations of columns of A.
2. $Ax = b \Rightarrow x = A^{-1}b$ if A is invertible.
3. The cyclic matrix C has no inverse. the column vectors of C are dependent and lie in the same plane.