① Recall:

Course Evaluation in eLearning

A symmetric matrix S is positive definite if and only if one of the following properties holds:

1. All n pivots of S are positive;
2. All n upper left determinants are positive;
3. All n eigenvalues of S are positive;
4. $x^T S x > 0$ for any $x \neq 0$;
5. $S = A^T A$ for a matrix A with independent columns.

§7.2 Bases and Matrices in the SVD = Singular Value Decomposition

Let A be an $m \times n$ matrix.

Question: Can we diagonize A?

Answer: singular value decomposition (SVD) of A.

$$A = U \Sigma V^T$$

$$= [u_1, \cdots, u_r, u_{r+1}, \cdots, u_m] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$u_1, \cdots, u_r$ is an orthogonormal basis for $C(A)$

$u_{r+1}, \cdots, u_m$ — — — $N(A^T)$

$v_1, \cdots, v_r$ — — — $C(A^T)$

$v_{r+1}, \cdots, v_n$ — — — $N(A)$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, σ_i's are singular values of A.

"A is digonalized" via $A v_1 = \sigma_1 u_1$, $A v_2 = \sigma_2 u_2$, $\cdots$ $A v_r = \sigma_r u_r$ (1)

$$A[v_1, \cdots, v_r] = [u_1, \cdots, u_r] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \quad (2)$$

$$A V_r = U_r \Sigma_r$$

③ Add $v_{r+1}, \cdots, v_n$ to V_r

$u_{r+1}, \cdots, u_m$ to U_r, we get

$$A[v_1 \cdots, v_r \cdots v_n] = [u_1 \cdots, u_r \cdots u_m]\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \end{bmatrix}$$

$$AV = U\Sigma$$

Since $V^{-1} = V^T$, we get ~~$A = U\Sigma V^T$~~

SVD $\quad A = U\Sigma V^T = u_1\sigma_1 v_1^T + \cdots + u_r\sigma_r v_r^T$

Ex 2. If $A = xy^T$ with unit vectors x and y, what is the SVD of A?

SVD of A is xy^T with singular value $\sigma_1 = 1$.

Proof of the SVD

Assume $A = U\Sigma V^T$

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$$
$$= V\Sigma^T U^T U\Sigma V^T$$
$$= V\Sigma^T\Sigma V^T$$

$\Sigma^T\Sigma$ is the eigenvalue matrix of A^TA.

each $\sigma^2 = \lambda(A^TA)$

V is the eigenvector matrix of A^TA

Now $Av_i = \sigma_i u_i$ gives unit vectors u_1 to u_r.

for $i \neq j$, $u_i^T u_j = \left(\frac{Av_i}{\sigma_i}\right)^T\left(\frac{Av_j}{\sigma_j}\right) = \frac{v_i^TA^TAv_j}{\sigma_i\sigma_j} = \frac{\sigma_j^2}{\sigma_i\sigma_j} v_i^Tv_j = 0$

Note: 1. $u_1, \cdots, u_r$ are eigenvectors of AA^T.

2. Let $v_{r+1}, \cdots, v_n$ be orthonormal bases for $N(A)$

$u_{r+1}, \cdots, u_m$ — — — $N(A^T)$

③ Then $A[v_1, \cdots v_r, v_{r+1}, \cdots, v_n] = [u_1, \cdots, u_r, u_{r+1}, \cdots, u_m]\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & \end{bmatrix}$

$$AV = U\Sigma \Leftrightarrow A = U\Sigma V^T$$

An Example of the SVD

Ex 3. Find U, Σ, V for $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$. the rank of A is 2.

$$A^TA = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \Rightarrow \sigma_1^2 = 45, \ \sigma_2^2 = 5$$

So, $\sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$

$(A^TA - \sigma_1^2 I)x = 0 \Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A^TA - \sigma_2^2 I)x = 0 \Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$Av_2 = \sigma_2 u_2 \Rightarrow u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$U = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Note: $A = U\Sigma V^T$

$= \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$

σ_1 is the maximum of the ratio $\|Ax\|/\|x\|$