① Recall:

1. Every ~~real~~ symmetric matrix S has real eigenvalues and perpendicular eigenvectors.
2. S is diagonalizable by $S = Q \Lambda Q^T$, where Q is an orthogonal eigenvector matrix Q.
3. For S, the signs of eigenvalues match that of the ~~pivots~~ pivots.

§ 6.5 Positive Definite Matrices.

~~positive~~ Symmetrice matrices that have positive eigenvalues are called positive define.

Question: 1. quick tests on positive definiteness of symmetric matrices? (avoid eigenvalues computation)

(2. Interesting application?)

When ~~do~~ does $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ have $\lambda_1 > 0$ and $\lambda_2 > 0$?

Theorem 1 eigenvalues of S are poositive if and only if $a > 0$ and $ac - b^2 > 0$

Ex: $S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is ~~po~~not positive definite since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -3 < 0$

$\lambda_1 = 3, \ \lambda_2 = -1$

$S_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$ is positive definite since $|a| = 1$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2 > 0$

$\lambda_1 = \frac{1}{2}(7 + \sqrt{41}), \ \lambda_2 = \frac{1}{2}(7 - \sqrt{41})$

Pf: $\cancel{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}$ Fact: the signs of eigenvalues match that of pivots for symmetric matrices.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xrightarrow{\text{Gauss}} \begin{bmatrix} a & b \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}$$

Thus, $\lambda_1 > 0, \lambda_2 > 0 \iff a > 0, \frac{ac-b^2}{a} > 0$

$$\iff a > 0, \; ac - b^2 > 0$$

Note: 1. Each pivot is a ratio of upper left determants.

2. Theorem 1 can be generalized to $n \times n$ case.

$S = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ is positive iff

1. $\det(a_{11}) > 0$
2. $\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0$
3. $\det S > 0$.

Ex. $S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is positive since $2 > 0$

$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$

$\det S = 4 > 0$

$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 4$

Energy-based Definition.

From $Sx = \lambda x$, we get

$$x^T S x = \lambda x^T x > 0 \quad \text{if } \lambda > 0, \; x \neq 0$$

Key point: $x^T S x > 0$ for any $x \neq 0$ if S is definite posive ($S = Q^T \Lambda Q$)

The number $x^T S x$ is the energy is many applications

Def 1. S is positive definite if $x^T S x > 0$ for any $x \neq 0$.

2 by 2 $\quad x^T S x = [x \; y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0$.

Prop 1. If S and T are positive definite, so is $S+T$.

Pf: $x^T (S+T) x = x^T S x + x^T T x > 0$ for any $x \neq 0$

Given a symmetric matrix S, $S = LDL^T = (L\sqrt{D})(L\sqrt{D})^T = A^T A$

③ Theorem 2. If columns of A are ~~independe~~ independent, then $S = A^T A$ is positive definite.

Pf: $x^T S x = x^T A^T A x$

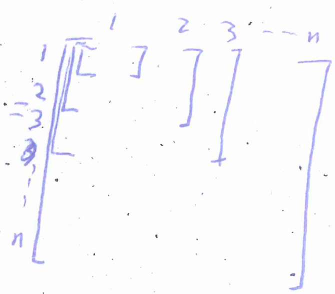
$= (Ax)^T A x$

$= \|Ax\|^2$

Since A has full column rank, $Ax \neq 0$ for any $x \neq 0$.

Theorem 3. S is positive definite if and only if

1. all n pivots of S are positive
2. all n upper left determinants are positive
3. ~~all~~ all n eigenvalues of S are positive.
4. $x^T S x > 0$ for any $x \neq 0$
5. $S = A^T A$ for A with independent columns.

$S =$ 

Ex 1. Test S and T for positive definiteness.

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

For S, $2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

$\det S = 4 > 0$

For T, $2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

$\det T = (1+b)(4-2b)$

~~de~~ T is positive if and only if $-1 < b < 2$.

(4) Test for a Minimum.

Fact : for $f(x)$, it takes the minimum at $x=0$ if

$$\frac{df}{dx}=0 \text{ and } \frac{d^2f}{dx^2}>0.$$

For $F(x,y)$, it takes a minimum $(x,y)=(0,0)$ if

1. $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$

2. $S = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}$ is positive definite.