①

Recall:

Let A be an $n \times n$ matrix.

1. Assume ~~If~~ A has n independent eigenvectors $x_1, \cdots, x_n$.
   Let $X = [x_1, \cdots, x_n]$. Then
   A is diagonalized by X $\quad X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

2. $A^k = X \Lambda^k \Lambda^{-1}$

3. A and C are similar if $A = BCB^{-1}$.
   Similar ~~ma~~ matrices have same eigenvalues.

§6.4 Symmetric Matrices

< Let S be a symmetric matrix.

Question: What are properties of eigenvalues for S?

Can we diagonilze S?

Spectral Theorem. Every symmetric matrix S has real eigenvalues in Λ and orthonormal eigenvectors in Q su chat.

$$S = Q\Lambda Q^{-1} = Q \Lambda Q^T$$

Ex 1.

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\det(S - \lambda I) = \lambda(\lambda - 5) \Rightarrow \lambda_1 = 0, \lambda_2 = 5$

$(S - 0I)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x_1 \in N(S)$

$(S - 5I)x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_2 \in C(S)$

Since S is symmetric, $C(S) = C(S^T)$

Thus, ~~x_2~~ x_1 and x_2 are perpendicular

Set $Q = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right]$.

② Then

$$Q^{-1} S Q = Q^T S Q = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}.$$

Real Eigenvalues ~~a~~ all eigenvalues of a real symmetric matrix are real.

> Let $\lambda = a + ib$ be a complex number. Its complex conjugate $\bar{\lambda} = a - ib$

Pf: Suppose $Sx = \lambda x$ (1)

Take conjugates of (1), get

$$S\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T S^T = \bar{x}^T S = \bar{x}^T \bar{\lambda} \quad (2)$$

~~Mult~~ Multiply ~~(1)~~ (1) by $\bar{x}^T$, get

$$\bar{x}^T S x = \lambda \bar{x}^T x \quad (3)$$

——— (2) by x, get

$$\bar{x}^T S x = \bar{\lambda} \bar{x}^T x \quad (4)$$

By (3) and (4), we have

$$(\lambda - \bar{\lambda}) \bar{x}^T x = 0 \Rightarrow \lambda = \bar{\lambda} \qquad Q.E.D.$$

Note: Since λ is real, the eigenvectors x is also real by solving $(S - \lambda I) x = 0$.

Orthogonal Eigenvectors eigenvectors of a real symmetric matrix are ~~a~~ perpendicular.

Pf: Suppose $Sx = \lambda_1 x$ and $Sy = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$.

Consider

$$\begin{aligned}(\lambda_1 x)^T y &= (Sx)^T y \\ &= x^T S^T y \\ &= x^T S y \\ &= x^T (\lambda_2 y)\end{aligned}$$

$$\Rightarrow (\lambda_1 - \lambda_2) x^T y = 0$$

$$\Rightarrow x^T y = 0$$

Ex 2. $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ has eigenvectors $x_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}$.

$$x_1^T x_2 = b(\lambda_2 - c) + (\lambda_1 - a) b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

③ Symmetric matrices S have orthogonal eigenvector matrices Q

$$S = Q \Lambda Q^T$$

$$= \begin{bmatrix} q_1 & \cdots, & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T$$

Complex Eigenvalues of Real Matrices

Let A be a real square matrices

$$Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$$

For real matrices, complex λ's and x's come in "conjugate pairs".

Ex 3. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda = \cos\theta + i\sin\theta$ and $\bar{\lambda} = \cos\theta - i\sin\theta$.

$$Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\Rightarrow A\bar{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \bar{\lambda} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note: $|\lambda| = 1$ holds for eigenvalues of every orthogonal matrix Q.

Fact 1: Let S be a real symmetric matrix. The number of positive eigenvalues of =

— — — positive pivots.

Ex: $S = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has pivots 1 and -8
eigenvalues 4 and -2.

Fact 2: all symmetric matrices are diagonalizable, even with repeated eigenvalues.