① Recall:

1. An eigenvector $x \neq 0$ of A satisfies:

$$Ax = \lambda x,$$ ~~where~~

The number λ is an eigen~~vector~~value of A.

2. An eigen~~vector~~value λ of A satisfies:

$$\det(A - \lambda I) = 0$$

3. The eigenvalues of A^2, A^{-1} are λ^2, λ^{-1}, with same eigenvectors.

4. The sum of eigenvalues:

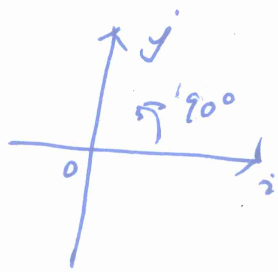
$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$$

The product of eigenvalues.

$$\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$$

5. Eigenvalues might ~~be~~ not be real numbers

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



§6.2 Diagonalizing a Matrix

Goal: Let A be an $n \times n$ matrix. Turn ~~the matrix~~ A into a diagonal matrix Λ when we use the eigenvectors properly.

Diagonalization Assume A has n independent eigenvectors $x_1, \dots, x_n$. Set $X = [x_1, \dots, x_n]$. Then

$$X^{-1} A X = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

where λ_i's are eigenvalues

②

Ex 1.

$$A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \text{ has eigenvalues } \lambda_1 = 1,\ \lambda_2 = 6$$

$$(A - I)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - 6I)x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = [x_1 \ x_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Why is $AX = X\Lambda$?

Assume $Ax_i = \lambda_i x_i$

Then $AX = A[x_1 \cdots x_n] = [\lambda_1 x_1 \cdots \lambda_n x_n]$

$$= [x_1 \cdots x_n]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$= X\Lambda$$

Note: Without n independent eigenvectors, we can't diagonalize.

Assume $A = X\Lambda X^{-1}$. Then

$$A^k = \underbrace{(X\Lambda X^{-1})(X\Lambda X^{-1})\cdots(X\Lambda X^{-1})}_{k}$$

$$= X\Lambda^k X^{-1}$$

Ex:

$$\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & 6^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}$$

Note: 1. If $\lambda_1, \cdots, \lambda_n$ are all distinct, then $x_1, \cdots, x_n$ are independent (I will give a proof later).

③ 2. We can multiply eigenvectors by nonzero constants.

$$Ax = \lambda x \Rightarrow A(cx) = \lambda(cx).$$

3. Some matrices have too few eigenvectors so that they can not be diagonalized (deeper reason will be ~~mention~~ mentioned later).

Ex: $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 \Rightarrow \lambda = 0$ (with multiplicity 2)

$Ax = 0x \Rightarrow x = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

no second eigenvector, A can not be diagonalized.

Theorem1 If $\lambda_1, \cdots, \lambda_n$ are all distant, then $x_1, \cdots, x_n$ are independent.

Pf: $n=2$. Assume $c_1 x_1 + c_2 x_2 = 0$

$A(c_1 x_1 + c_2 x_2) = 0$

$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$ ①.

$\lambda_2 (c_1 x_1 + c_2 x_2) = 0$

$c_1 \lambda_2 x_1 + c_2 \lambda_2 x_2 = 0$ ②

① − ②, get

$(\lambda_1 - \lambda_2) c_1 x_1 = 0 \xRightarrow{\lambda_1 \neq \lambda_2} c_1 x_1 = 0 \Rightarrow c_1 = 0$

Thus, $c_2 = 0$.

Note: An n by n matrix with n different eigenvalues must be diagonalizable

Ex2. The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .5$.

④ b

$$A = X \Lambda X^{-1}$$

$$\Leftrightarrow \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$$

Thus, $A^k = X \Lambda^k X^{-1}$

$$= \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$$

let $k \to \infty$, $A^{\infty} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$

$$= \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$$

Similar Matrices: Same Eigenvalues.

$A = X \Lambda \cancel{B} X^{-1}$ A has same eigenvalues as Λ

Let $A, \cancel{B}\, C$ be $n \times n$ matrices. We call A and $\cancel{B}C$ are similar if $A = BCB^{-1}$ for some invertible matrix B

Theorem 2. Similar matrices have same eigenvalues.

Pf. Assume $A = BCB^{-1}$, and $Cx = \lambda x$. Then

$$\begin{aligned} A(Bx) &= (BCB^{-1})(Bx) \\ &= BCx \\ &= B(\lambda x) \\ &= \lambda(Bx) \end{aligned}$$

Ex: $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ or any $D = \dfrac{xy^T}{x^Ty}$

$\downarrow$

projection matrices

(mention a bit about other sections).