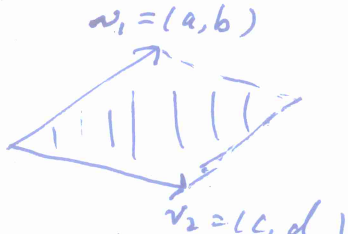
① Recall:

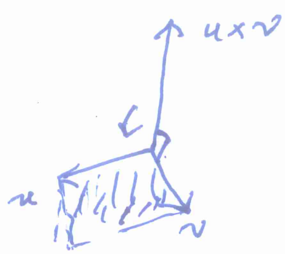
1. Cramer's Rule solves $Ax=b$ by $x_i = |B_i| / |A| = \dfrac{|a_1 \cdots \overset{i}{b} \cdots a_n|}{|A|}$

2. Let C be the cofactor matrix of A. Then $A^{-1} = \dfrac{C^T}{\det A}$

3. The volume of a box is $|\det A|$, when the box edges are rows of A.



4. In $\mathbb{R}^3$, the cross product $u \times v$ is perpendicular to u and v.



Chapter 6 Eigenvalues and Eigenvectors

§6.1 Introduction to Eigenvectors.

Let A be a square matrix.

Consider the difference equation

$$u_{k+1} = A u_k$$

Given u_0, what is u_{2019}?

$$u_{2019} = A^{2019} u_0$$

Question: how to compute A^{2019}?

Answer: eigenvalues of A.

(First explain eigenvectors)

Certain nonzero vectors x are in the same direction as Ax. Those are eigenvectors.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

②

$\overset{\neq 0}{(A-\lambda I)x = 0}$ Fact: $Bx = 0$ has no zero solution

$\Rightarrow \det(A-\lambda I) = 0$ $\Leftrightarrow \det(B) = 0.$

Ex 1. Let $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$

Consider $\det(A - \lambda I) = 0$

$$\begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$
$$= (\lambda-1)(\lambda-\tfrac{1}{2})$$

A has two eigenvalues $\lambda = 1$ and $\lambda = \frac{1}{2}$

$(A-I)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

$(A-\frac{1}{2}I)x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A^2 x_1 = x_1$

$A^2 x_2 = (\frac{1}{2})^2 x_2$

When A is squared, the eigenvalues are squared. Then eigenvectors are the ~~same~~ same.

Consider

$u_{n+1} = A u_n$ with $u_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$

What is $u_{2019} = A^{2019} u_0$?

By solving linear equations, we have

$$u_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + (.2) x_2.$$

Thus, $u_{2019} = A^{2019} u_0$

$$= A^{2019}[x_1 + (.2) x_2]$$
$$= A^{2019} x_1 + (.2) \cdot A^{2019} x_2$$
$$= x_1 + (.2)(\tfrac{1}{2})^{2019} x_2$$

$u_{2019} \approx x_1$

Note: 1° ~~A~~ In Ex 1, A is a Markov matrix, whose sum of each column is equal to 1.

2° Please read Ex 1~2 in the textbook.

The equation for the eigenvalues.

Let A be an $n \times n$ square matrix.

Eigenvalue λ is an eigenvalue of A $\Leftrightarrow$ $A - \lambda I$ is ~~sigul~~ singular.

Equation for ~~the~~ eigenvalues $\det(A - \lambda I) = 0$.

Note: $\det(A - \lambda I)$ is the characteristic polynomial of degree n for A. ~~We~~ The roots of it are eigenvalues of A.

For each eigenvalue λ solve $(A - \lambda I)x = 0$ to find an eigenvector x.

Ex 4. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find its λ's and x's.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 5\lambda$$
$$= \lambda(\lambda - 5) \Rightarrow \lambda = 0, \lambda = 5$$

$(A - 0I)x = 0 \Rightarrow x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(A - 5I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (mention the summary in ~~the~~ page 293 of the textbook)

Determinant and Trace

~~Ex 1~~: Let A be an $n \times n$ square matrix.

The sum of entries along the diagonal is the trace of A.

Fact 1: the sum of eigenvalues of A is equal to the trace of A

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}.$$

Fact 2: the product of eigenvalues of A is equal to the determinant of A.

$$\lambda_1 \cdots \lambda_n = \det(A).$$

Note: we can use Fact 1~2 to compute eigenvalues of 2×2 matrices.

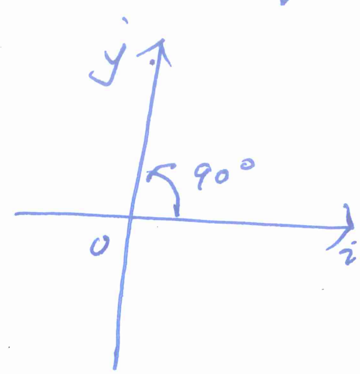
Ex: $A = \begin{bmatrix} 1 & 9 \\ 0 & 2 \end{bmatrix}$

$$\begin{cases} \lambda_1 \cdot \lambda_2 = 1 \times 2 = 2 \\ \lambda_1 + \lambda_2 = 1 + 2 = 3 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 3 - \lambda_2 \\ (3-\lambda_2)\lambda_2 = 2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

Imaginary Eigenvalues

(Since we need to solve a polynomial to get eigenvalues of a matrix), the eigenvalues might not be real numbers

Ex 5. The 90° rotation $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues



$$\det(Q - \lambda I) = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = i, -i$$

$$(Q - iI)x = 0 \Rightarrow x = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$(Q + iI)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ i \end{bmatrix}$$