① Recall:

Office: FO 2.106

1. Pivot Formula:

$$A \xrightarrow{\text{Gauss}} U$$

$\det A = \pm$ (product of pivots).

2. Big Formula: (derived from rules 1~3)

$\det A =$ sum over $n!$ column permutations $P = (\alpha, \beta, \cdots, \omega)$

$$= \sum (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

3. Cofactor Formula: (useful if A has many zeros in a row)

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

§5.3 Cramer's Rule, Inverses, and Volumes.

Cramer's Rule solves $Ax = b$ (by determinants).

key idea:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

Then $\det A \cdot \cancel{\det} \begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = \det B_1$

$\Leftrightarrow \det A \cdot x_1 = \det B_1$

if $\det A \neq 0$, then $x_1 = \dfrac{\det B_1}{\det A}$.

(Similarly, we compute x_2)

Assume $A = [a_1 \ a_2 \ a_3]$.

$$[a_1 \ a_2 \ a_3] \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = [a_1 \ b \ a_3] = B_2$$

$\Rightarrow \det A \cdot x_2 = \det B_2 \Rightarrow x_2 = \dfrac{\det B_2}{\det A}$.

Ex 1. Consider

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -2, \quad \det B_1 = \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} = -4, \quad \det B_2 = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 2$$

Thus, $x_1 = \dfrac{\det B_1}{\det A} = 2$, $x_2 = \dfrac{\det B_2}{\det A} = -1$

Cramer's Rule Assume $\det A \neq 0$. Then $Ax = b$ is solved by:

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \quad \dots, \quad x_n = \frac{\det B_n}{\det A}$$

B_j has jth column of A replaced by b.

(Note: Cramer's Rule is not a practical method to solve linear equations, but is useful for theoretical analysis. It gives an explicit formula for the solution x.)

Ex 2. Use Cramer's Rule to compute inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Assume $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$$AA^{-1} = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The determinants are

$$\underbrace{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}_{= |A|}, \quad \begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}, \quad \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}, \quad \begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}$$

③ Thus, $x_1 = \frac{d}{|A|}$, $x_2 = \frac{-c}{|A|}$, $y_1 = \frac{-b}{|A|}$, $y_2 = \frac{a}{|A|}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note: A^{-1} involves the cofators of A.

$n=3$. Solve

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ to get column 1 of } A^{-1},$$

By Cramer's Rule, $|B_j|$'s are

$$\underbrace{\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}}_{= C_{11}} \quad \underbrace{\begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}}_{= C_{12}} \quad \underbrace{\begin{vmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}}_{= C_{13}}$$

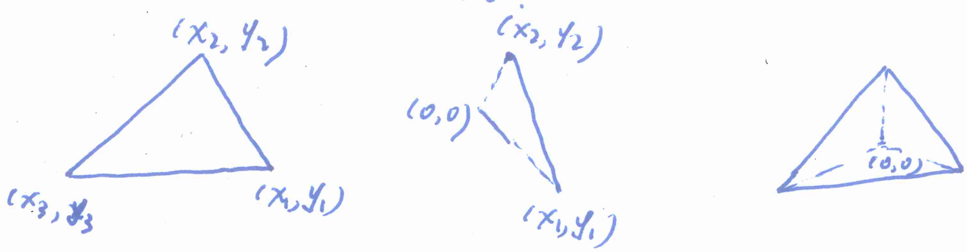
~~FORM~~ Formular for A^{-1}

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^T}{\det A}, \text{ where } C = (C_{ij})$$

Note: the formular for A^{-1} can also be proved by cofactor formulas. (please check the textbook).

Area of a ~~Tea~~ Triangle

Question; Given corners (x_1, y_1), (x_2, y_2), (x_3, y_3) of a triangle, what is the area?

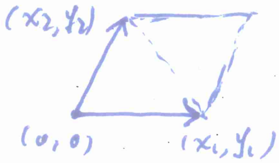


Answer: area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ Area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ where $(x_3, y_3) = (0,0)$

reduce to three special triangles from $(0,0)$:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{array}{l} +\frac{1}{2}(x_1 y_2 - x_2 y_1) \\ +\frac{1}{2}(x_2 y_3 - x_3 y_2) \\ +\frac{1}{2}(x_3 y_1 - x_1 y_3). \end{array}$$

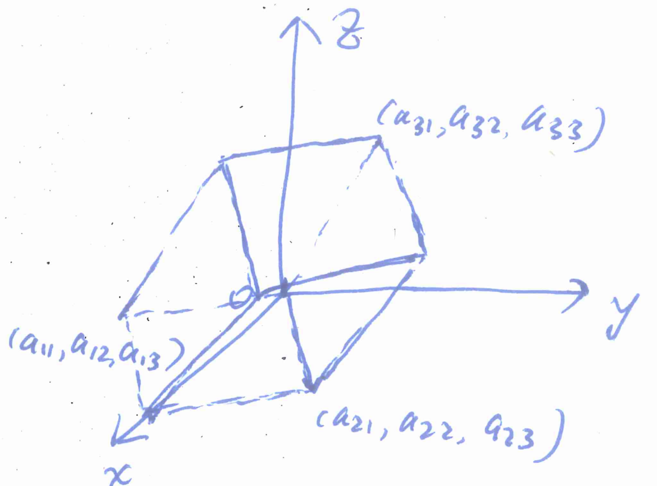
Conside the case $(x_3, y_3) = (0, 0)$.



a parallelogram starting from $(0,0)$ has area $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

~~Schetch~~ sketch of proof: the area has the same properties 1~3 as the determinant.

$n = 3$



volume of box

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The Cross Product

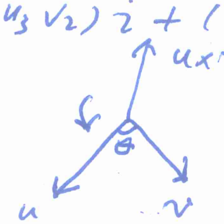
Def The cross product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is a vector

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\, i + (u_3 v_1 - u_1 v_3)\, j + (u_1 v_2 - u_2 v_1)\, k$$

where $i = (1, 0, 0)$

$j = (0, 1, 0)$

$k = (0, 0, 1)$



$u \times v$ is perpendicular to u and v and $v \times u = -(u \times v)$.

⑤ By sign reversal of determinants, $v \times u = -(u \times v)$

$$u \cdot (u \times v) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

Property 3 $u \times u = 0$ (By properties of determinant)

When u and v are parallel, $u \times v = 0$, Moreover,

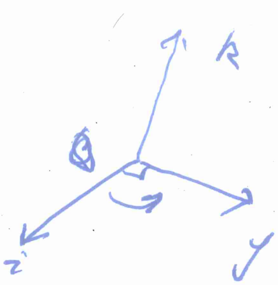
$$\|u \times v\| = \|u\| \|v\| \cdot |\sin\theta|$$

$\Rightarrow$ the ~~long~~ length of $u \times v$ is the ~~p~~area of the parallelogram with sides u and v.

Ex 9. $i = (1, 0, 0)$

$j = (0, 1, 0)$

$k = (0, 0, 1)$



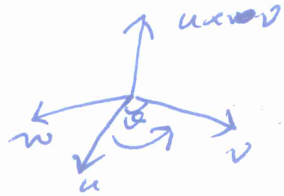
right hand rule

Then $k = i \times j$

Note: the direction of $u \times v$ is determined by right hand rule.

Triple Product = Determinant = Volume

Given vectors u, v and w



Triple product

$$(u \times v) \cdot w = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$(u \times v) \cdot w = 0 \Leftrightarrow u, v, w$ lie in the same plane.

$(u \times v) \cdot w$ = volume of the box with sides u, v, w.