① Recall:

1. The determinant is defined by $\det I = 1$, sign reversal, and linearity in each row.
2. After elimination $\det A$ is $\pm$(product of the pivots)
$$A \xrightarrow{\text{Gauss}} U$$
3. A is invertible $\Leftrightarrow \det A \neq 0$
4. $\det AB = (\det A)(\det A)$
$\det A^T = \det A$

§5.2 Permutations and Cofactors

determinant $\begin{cases} \text{pivots} \checkmark \\ \text{big formula} \\ \text{cofactors} \end{cases}$

(Let us first review the pivot approach)

The pivot formula

Let A be a square matrix.

$PA = LU$, with $U = \begin{bmatrix} d_1 & & * \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix}$

Then $\det P \cdot \det A = \det L \cdot \det U$

$\Rightarrow \det A = \pm(d_1 d_2 \cdots d_n)$.

Ex 1.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

② Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$$PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \det A = -4 \times 2 \times 1 = -8$.

Ex 2.

— tridiagonal matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \ddots & \ddots & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \ddots & \ddots \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

$\det A = 2 \times \frac{3}{2} \times \frac{4}{3} \times \cdots \frac{n+1}{n} = n+1$.

The Big Formula for Determinants

Idea: Use rules 1~3, linearity, sign reversal and $\det I = 1$ to derive a explicit formulas for determinant (directly from the entries a_{ij}).

Why Big?

The formulas has $n!$ terms

$n! = 1 \times 2 \times 3 \times \cdots \times n$.

(When n increase, $n!$ increase ~~dramatical~~ very fast.

3 For $n=3$, there are $3! = 3\times 2\times 1 = 6$ terms.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = \begin{matrix} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{matrix}$$

Note: 1° Each term like $a_{11}a_{23}a_{32}$ has one entry from each row, and also one entry from each column.

2° There ~~are~~ is a sign for each term.

How to derive the big formula?

$n=2$,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$[a\ b] = [a\ 0] + [0\ b]$ (break into two parts)

$[c\ d] = [c\ 0] + [0\ d]$

By linearity,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \quad \text{(break ~~down~~ up row 1)}$$

$$= \underset{=0}{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underset{=0}{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}} \quad \text{(break ~~d~~ up row 2)}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

– permutation matrix with column order ~~(2,1)~~ $(2,1)$

$$= ad - bc$$

$n=3$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

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$$+\begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

Note: 1° For each determinant, a_{ij}'s come from different columns (otherwise, there will be two identical ~~co~~ columns)

2° The six permutation matrix has ~~the~~ column numbers:

$(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(2,1,3)$, $(3,2,1)$

$$\det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} \cancel{+a_{13}a_{21}a_{32}}$$

$$+a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}$$

$$+a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} \cancel{+a_{13}a_{22}a_{31}}$$

(~~We can determine~~ Column numbers determine signs of permutation matrix, and are encoded by column ~~index~~ indices of each term).

$n\times n$ case.

$\det A =$ sum over all $n!$ column permutations $P=(\alpha,\beta,\cdots,w)$

$= \sum \det(P)\, a_{1\alpha}a_{2\beta}\cdots a_{nw}$ — BIG FORMULA.

Ex 4. Consider

$$Z = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix}$$

The ~~on~~ only non-zero term comes from the diagonal. So,

$Z = \cancel{0} + 1\times 1\times c\times 1 = c.$

⑤

Ex 5.

$$|A| = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = |P|$$

P has column number, $(2, 1, 4, 3)$ (it has even number of exchanges).

$\det P = +1. \Rightarrow \det A = 1.$

Determinant by Cofactors.

3×3 case

(We group the big formula by first row entries)

$$\det A = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{\text{cofactor } C_{11}} + a_{12}\underbrace{(a_{23}a_{31} - a_{21}a_{33})}_{C_{12}} + a_{13}\underbrace{(a_{21}a_{32} - a_{22}a_{31})}_{C_{13}}$$

Cofactors are 2×2 determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

We cross out row 1 and column j to get a submatrix M_{1j} of size 2.

$$C_{1j} = (-1)^{1+j} \det M_{1j}$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Note: 1° cofactor expansion works for any row i

2° ——— ——— any ~~row~~ column j.

⑥ Cofactor Formula $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

Cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$

((Cofactors are useful when matrices have many zeros)

Ex 6.

$$\underbrace{\begin{vmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix}}_{D_4} = 2\overbrace{\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix}}^{D_3} - (-1)\begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix}$$

$$= 2D_3 - (-1)(-1)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 2D_3 - D_2$$

Direct computation gives $D_2 = 3$, $D_3 = 4$.

Thus, $D_4 = 5$.