① Recall:

1. vectors $q_1, \cdots, q_n$ are orthonormal if

$$\begin{cases} q_i^T q_j = 0 & \text{if } i \neq j \\ q_i^T q_j = 1 & \text{if } i = j \end{cases}$$

Let $Q = [q_1, \cdots, q_n]$, then $Q^T Q = I$.

2. If Q is square, then $Q^T = Q^{-1}$

3. If Q has orthonormal columns, $\|Qx\| = \|x\|$

4. projection onto $C(Q)$ is $P = QQ^T$

5. If Q is square, then $P = QQ^T = I$ and

$b = q_1(q_1^T b) + \cdots q_n(q_n^T b)$.

6. Let a, b, c be independt vectors.

$a, b, c \xrightarrow{\text{Gram-Schmidt}} q_1, q_2, q_3$ orthonormal bases.

$$A = [a\ b\ c] = [q_1\ q_2\ q_3]\begin{bmatrix} * & \times & \times \\ & * & \times \\ & & * \end{bmatrix}$$

$$= QR$$

Chapter 5

Determinants

~~§5.1 The Properties of Determinants.~~

Determinant is a function of square matrix.

$$f: \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$
$$A \longmapsto |A|$$

A has inverse ~~if and~~ $\Leftrightarrow$ ~~determinant~~ $|A| \neq 0$

Determinants gives formulas for A^{-1} and $A^{-1}b$ (Cramer's Rule)

- - - - is the volume of a box ~~with~~ whose edges are rows of A.

§5.1 The Properties of the Determinant

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(Rules ~~1 ~ 3 determine gives the determinant~~ determinant)

(~~Rules 1 ~ 3 ⇒ Rules 4-10~~).

(We will check all rules with 2×2 matrix, but rules apply to n×n matrix).

1. $|I| = 1$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

2. The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad.$$

Ex: Let P be a permutation matrix. Then $\det P = \pm 1$

3. The determinant is a linear function of each row.

1° $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2° $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Ex. $\begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$, $\begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

③

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2^2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2^2$$

$$\begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = t^2$$

< Note: 1° Rules 1~3 gives the determinant.
2° Rules 1~3 ⇒ Rules 4-10

4. If two rows of A are equal, then $\det A = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

Set $D = \begin{vmatrix} a & b \\ a & b \end{vmatrix}$. By rule 2, $D = -D \Rightarrow D = 0$

5. Doing one-step elimination leaves $\det A$ unchanged

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

By rule 3, $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$A \xrightarrow{\text{Gauss}} U$$

$$\det A = \pm \det U$$

6. A matrix with zero rows has $\det A = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0, \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

By rule 5, $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} = 0$

7. If A is triangular then $\det A = a_{11} a_{22} \cdots a_{nn}$

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad \text{and} \quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

Assume $a_{11} a_{22} \cdots a_{nn} \neq 0$

$$A = \begin{bmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{bmatrix} \xrightarrow{\text{Gauss}} \begin{bmatrix} a_{11} & & \\ & a_{22} & 0 \\ 0 & & \ddots \\ & & & a_{nn} \end{bmatrix}$$

$$\det A = \cancel{\det} \begin{vmatrix} a_{11} & & \\ & a_{22} & 0 \\ 0 & & \ddots \\ & & & a_{nn} \end{vmatrix} \overset{\text{Rule 3}}{=\!=\!=} a_{11} a_{22} \cdots a_{nn}$$

if ~~there~~ some $a_{ii} = 0$, $\det A = 0$

8. A is invertible $\Leftrightarrow$ $\det A \neq 0$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is singular $\Leftrightarrow$ $ad - bc = 0$.

Assume that $a \neq 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{Gauss}} \begin{bmatrix} a & b \\ 0 & d - (c/a)b \end{bmatrix}$$

(A is singular $\Leftrightarrow$ $ad - bc = 0$).

9. $|AB| = |A||B|$

Sketch of Pf: if $|B| \neq 0$,

Set $D(A) = |AB| / |B|$ Then $D(A)$ ~~show~~ satisfy rules $1 \sim 3$

$\Rightarrow D(A) = |A|$.

if $|B| = 0$, then AB is singular $\Rightarrow |AB| = 0$.

10. $\det A = \det A^T$

if A is singular, then A^T is also singular

$\det A = \det A^T = 0$

Otherwise, $PA = LU$

(5)

$$A^T P^T = U^T L^T$$

Thus, $\det P \det A = \det L \det U$

$\det A^T \det P^T = \det U^T \det L^T$

$\det L^T = \det L = 1$

$\det U = \det U^T$

$PP^T = I \Rightarrow \det P = \det P^T = \pm 1$

Therefore, $\det A = \det A^T$.

Note: rules for rows also apply to columns of A.