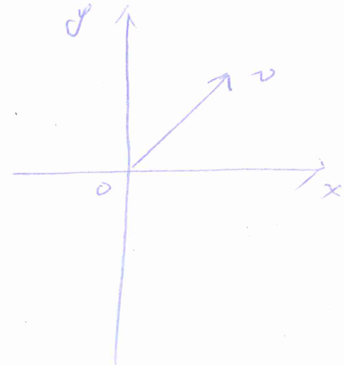
Recall:

1. a vector in 2-D space:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



2. $v + w = (v_1 + w_1, v_2 + w_2)$

   $cv = (cv_1, cv_2)$

3. a linear combination of u, v and w is $cu + dv + ew$.

4. Let u, v, w be vectors in 3-D space.

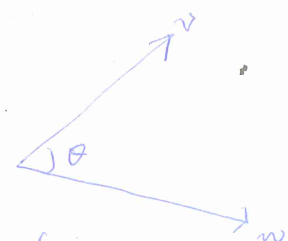
   Take all linear combinations of

(Typically) 1° $cu \longleftrightarrow$ a line through $(0, 0, 0)$

2° $cu + dv \longleftrightarrow$ a plane ——— ———

3° $cu + dv + ew \longleftrightarrow$ the whole 3-D space.

§1.2 Lengths and Dot Products



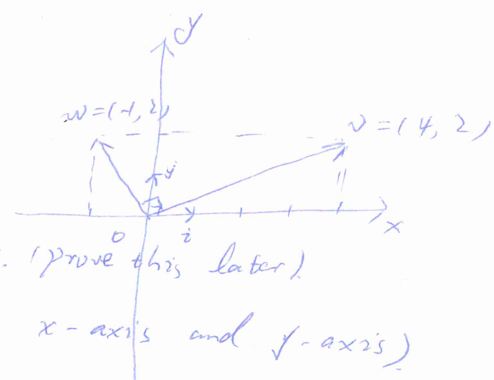
dot product of $v = (v_1, v_2)$ and $w = (w_1, w_2)$

$$v \cdot w = v_1 w_1 + v_2 w_2.$$

(geometric meaning: cosines of angles between v and w's and lengths of vectors)

Ex 1. $v = (4, 2)$, $w = (-1, 2)$

$$v \cdot w = 4 \times (-1) + 2 \times 2 = 0$$



dot product is zero $\Longleftrightarrow$ perpendicular vectors. (prove this later)

$i = (1, 0)$, $j = (0, 1)$ (unit vectors along x-axis and y-axis)

$i \cdot j = 1 \times 0 + 0 \times 1 = 0$

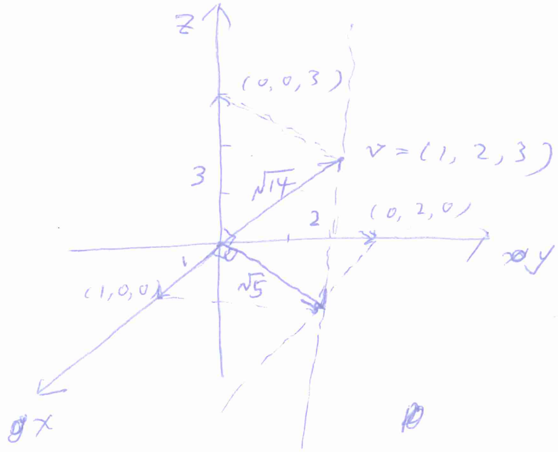
Note: 1. $w \cdot v = v \cdot w$

2. $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_n)$

   $v \cdot w = v_1 \cdot w_1 + v_2 w_2 + \cdots + v_n w_n$.

② Lengths and Unit vectors.

(Let us start from an example)



$v = (1, 2, 3)$

$v \cdot v = 1^2 + 2^2 + 3^2$

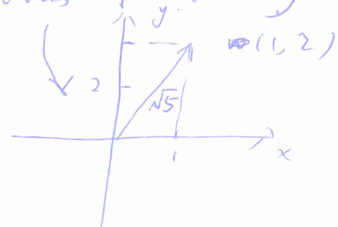
$= 14$

~~$\|v\|$~~

the length of v is

$$\|v\| = \sqrt{v \cdot v} = \sqrt{14}$$

~~$w = (w_1, w_2), \|w\|^2 = w_1^2 + w_2^2$~~ [pi'θægərəs] (Pythagoras formula)



Def~~inition~~ 1 The length $\|v\|$ of a vector $v = (v_1, \cdots, v_n)$ is:

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Def 2 A unit vector u is a vector with ~~leg~~ length 1, i.e., $u \cdot u = 1$

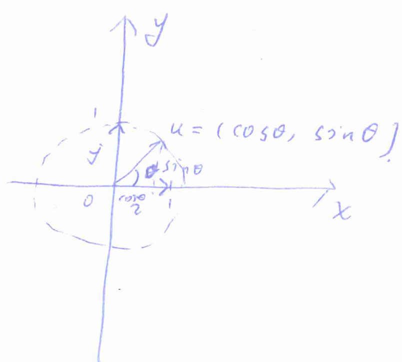
Ex 4.

standard unit vectors

unit vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$u \cdot u = \cos^2\theta + \sin^2\theta = 1.$



Let $v = (2, 2, 1)$. Set

$$u = \frac{v}{\|v\|} = \frac{1}{3}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \quad \text{unit vectors}$$

~~Unit~~ (vector $u = v / \|v\|$ is a unit vector in the same direction as v)

The angle between two vectors (geometric meaning of dot product)

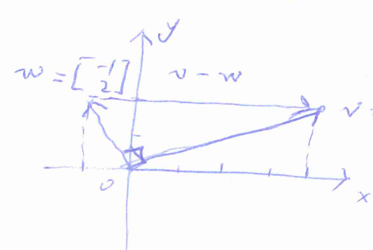
Theorem $v \cdot w = 0 \Leftrightarrow v$ is perpendicular to w. hypotenuse [hai'pɑ:tənu:s]

Pf for 2-D case.

"$\Rightarrow$": ~~If As~~ By Pythagoras Law,

$$\|v\|^2 + \|w\|^2 = \|v - w\|^2$$

$\Leftrightarrow$



③ $(v_1^2+v_2^2)+(w_1^2+w_2^2) = (v_1-w_1)^2+(v_2-w_2)^2$

$\Leftrightarrow$ ~~$v_1 w_1 + v$~~ $0 = -2(v_1 w_1 + v_2 w_2)$

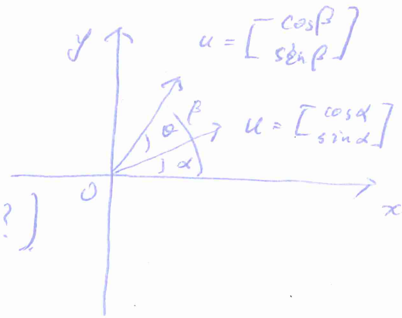
$\Leftrightarrow$ $v\cdot w = 0$

"$\Leftarrow$": HW. $\square$

(Note: the dot product reveal the exact angle θ between two vectors.)

Unit vectors u and U at angle θ have $u\cdot U = \cos\theta$. $(|u\cdot U| \le 1)$

$$u\cdot U = \cos\alpha\cdot\cos\beta + \sin\alpha\sin\beta$$
$$= \cos(\beta-\alpha)$$
$$= \cos\theta$$

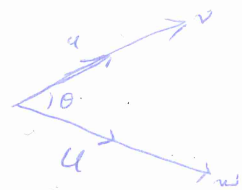


[What if v and w are not unit vectors?]

$u = \frac{v}{\|v\|}$, $U = \frac{w}{\|w\|}$.

cosine formula if v and w are nonzero, then $\frac{v\cdot w}{\|v\|\,\|w\|} = \cos\theta$

Since $|\cos\theta| \le 1$, we have



Schwarz Inequality $|v\cdot w| \le \|v\|\,\|w\|$

Triangle Inequality $\|v+w\| \le \|v\| + \|w\|$

Ex 5. Let $v = \begin{bmatrix}2\\1\end{bmatrix}$ and $w = \begin{bmatrix}1\\2\end{bmatrix}$, compute $\cos\theta$ and check both inequalities

$v\cdot w = 2\times1 + 1\times2 = 4$.

$\|v\| = \sqrt{2^2+1^2} = \sqrt5$

$\|w\| = \sqrt5$

$\cos\theta = \frac{v\cdot w}{\|v\|\cdot\|w\|} = \frac45$

$v\cdot w = 4 < \|v\|\cdot\|w\| = 5$

$v+w = (3,3)$

$\|v+w\| = \sqrt{3^2+3^2} = 3\sqrt2$

$\Rightarrow \|v+w\| = \sqrt{18} < \|v\|+\|w\| = 2\sqrt5$
$= \sqrt{50}$

④ Review

1. $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_n)$, $v \cdot w = v_1 w_1 + \cdots + v_n w_n$

2. $\|v\| = \sqrt{v \cdot v}$ unit vector $u = \cancel{\frac{v}{\|v\|}} \frac{v}{\|v\|}$.

3. $v \cdot w = 0 \iff$ v and w are perpendicular.

4. $\cos\theta = \dfrac{v \cdot w}{\|v\|\|w\|}$

Schwarz inequality $|v \cdot w| \le \|v\| \cdot \|w\|$.