① Recall:

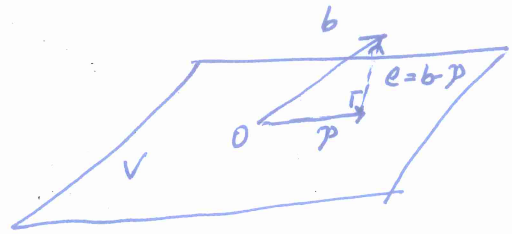
1. The projection of b onto the line through a is

$$p = a\hat{x} = a(a^Tb/a^Ta)$$

Projection matrix $P = aa^T/a^Ta$

2. Projecting b onto a subspace leaves $e = b - a$ perpendicular to the subspace.

3. When A has full rank n, $A^TA\hat{x} = A^Tb$ $\cancel{AA^T\hat{x} = A^Tb}$ gives $\hat{x}$ and $p = A\hat{x}$



projection matrix $P = A(A^TA)^{-1}A^T$ has $P^T = P$, $P^2 = P$, and

$Pb = p$.

§4.3 Least Squares Approximations.

Let A be a $m \times n$ matrix

When $m > n$, then it often happens

$Ax = b$ has no solution.

Consider the error $e = b - Ax$

Question: Is there x such that

$\|e\|^2$ is as small as possible? Answer: $\hat{x}$ is a solution of $A^TA\hat{x} = A^Tb$.

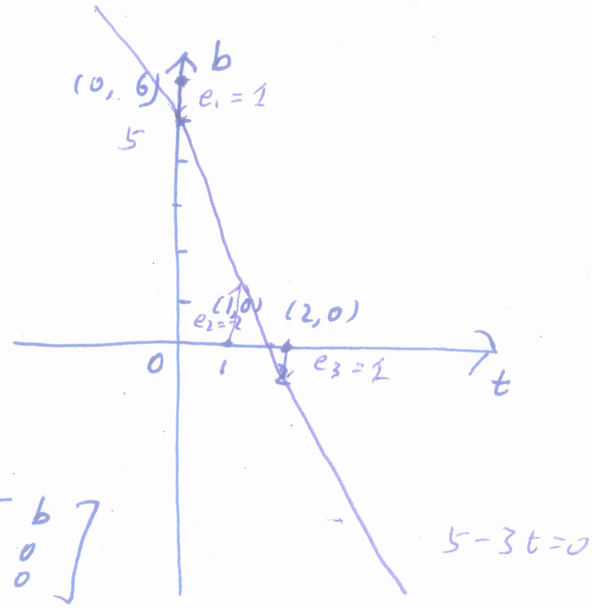
Application: fitting a straight line to m points.

Ex 1. Find the closest line to ~~th~~ points $(0, 6)$, $(1, 0)$, and $(2, 0)$

↓

the sum of vertical errors to each point is minimal.

② Assume the line $b = C + Dt$ goes through those points. Then



$$\begin{cases} C + D\cdot 0 = 6 \\ C + D\cdot 1 = 0 \\ C + D \cdot 2 = 0 \end{cases}$$

$\Leftrightarrow$ $Ax = b$ (*), where

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, ~~b~~ $x = \begin{bmatrix} C \\ D \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

(*) has no solution. Consider the error $e = b - Ax$

~~The best line with smallest vertical errors~~ Find $\hat{x}$ such that $\|e\|^2$ is minimal.

$\Rightarrow A^T A \hat{x} = A^T b \Rightarrow \hat{x} = (5, -3)$

So, $5 - 3t = 0$ is the best line for the 3 points.

Minimizing the Error

Let A be a $m \times n$ matrix, $b \in \mathbb{R}^n$

~~Geometry each Ax belongs to C(A). We are looking for~~

Question: Given A, b, find $\hat{x}$ such that

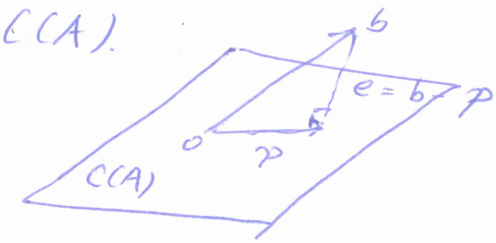
$\|e\|^2 = \|b - A\hat{x}\|^2$ is minimal

By Geometry each Ax belongs to $C(A)$. We are looking for $p = A\hat{x}$ such that p is closest to b.

Answer: p is the projection of b onto $C(A)$.

Thus, $e = b - A\hat{x}$ is orthogonal to $C(A)$.

$\Rightarrow A^T A \hat{x} = A^T b$.



By calculus

Consider the error function

$E = \|Ax - b\|^2$

③ E is minimal $\Rightarrow \frac{\partial E}{\partial x_i} = 0$, $i = 1, \cdots, n$.

$$\Updownarrow$$

$$A^T A \hat{x} = A^T b$$

Fitting a Straight line

Give m points $(t_1, b_1), \cdots, (t_m, b_m)$.

The best line $C + Dt$ misses the points by vertical distances $e_1, \cdots, e_m$. We want minimizes $E = e_1^2 + \cdots + e_m^2$

A line goes through the m points gives

~~$Ax = b$~~

~~where~~

$$\begin{cases} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{cases}$$

$\Leftrightarrow \quad Ax = b$ (*) with $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}$

(In general) (*) has no solution

Find $\hat{x}$ such that

$E = \|b - Ax\|^2$ is minimal

$\Rightarrow A^T A \hat{x} = A^T b$

$A^T A = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}$

$A^T b = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$

$A^T A \hat{x} = A^T b \Leftrightarrow \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$

Ex 2. Given 3 points $(-2,1)$, $(0,2)$, $(2,4)$

Find the best line $C+Dt$ fitting those 3 points.

Assume a line $C+Dt$ goes through 3 points.

$$\begin{cases} C + D(-2) = 1 \\ C + D(0) = 2 \\ C + D(2) = 4 \end{cases} \Leftrightarrow Ax = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$Ax=b$ has no solution.

Find $\hat{x}$ such that $\|b - A\hat{x}\|^2$ is minimal

$$\Rightarrow A^TAx = A^Tb \Leftrightarrow \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{cases} C = \frac{7}{3} \\ D = \frac{6}{8} \end{cases}$$