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Recall:

1. V and W are orthogonal if $v \cdot w = 0$ for each $v \in V$ and $w \in W$.

2. V and W are orthogonal complements if W contains all vectors perpendicular to V. In R^n, dimensions of V and W add to n.

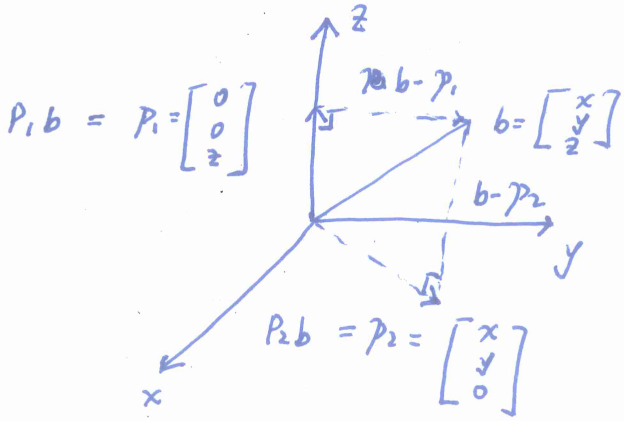
3. $N(A)$ and $C(A^T)$ are orthogonal complements
$N(A^T)$ $C(A)$ } Fundamental Theorem of Linear Algebra Part 2.

4. Any n independent vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$.
Any n spanning vectors are independent.

for each $x \in \mathbb{R}^n$, $x = x_n + x_r$, where $x_n \in N(A)$, $x_r \in C(A^T)$

(Question: how to make such a splitting?)

§4.2 Projections.



Question: 1. What are projections of $b = (x, y, z)$ onto the z axis and the xy plane?

2. What matrices P_1 and P_2 produce those projections onto a line and a plane?

Answer: 1. $p_1 = (0, 0, z)$ a vector along z axis

$p_2 = (x, y, 0)$ a vector in xy plane.

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2. $p_1 = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{P_1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$p_2 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Note:

1° z-axis and xy-plane are orthogonal complements

$b = p_1 + p_2$,

$\Rightarrow P_1 + P_2 = I$

2° $P_1^2 = P_1$, $P_2^2 = P_2$ - projection matrices

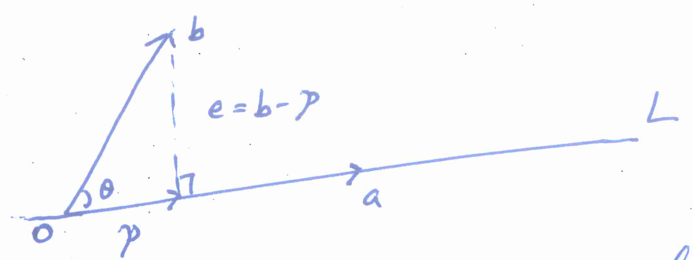
Goal: Given a subspace V in $\mathbb{R}^n$, compute ~~its~~ ~~for each $b \in \mathbb{R}^n$~~, projection matrix P such that for each $b \in \mathbb{R}^n$, the projection onto V is $p = Pb$.

Projection Onto a Line.

Given a line L through the origin in direction of $a = (a_1, \cdots, a_m)$

Find a vector p along L s.t. p is closest to $b = (b_1, \cdots, b_m)$

i.e. the error $b - p$ is perpendicular to a. (key point for projection)



(Let us calculate p by using algebra)

Set $p = \hat{x} a$.

Since $a \cdot (b - p) = 0$, we have

$a \cdot (b - \hat{x} a) = 0$

$a \cdot b - \hat{x}\, a \cdot a = 0$

we have $\hat{x} = \dfrac{a \cdot b}{a \cdot b} = \dfrac{a^T b}{a^T a}$

Thus, the projection of b onto the line L is

$$p = \hat{x} a = \frac{a^T b}{a^T a} a$$

Note: 1. If $b = a$, then $\hat{x} = 1$

2. If $b \cdot a = 0$, the $a^T b = 0$

Ex 1 Project $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$a^T \cdot b = 5$, $a^T a = 9$

$$\Rightarrow p = \frac{a^T b}{a^T a} a = \frac{5}{9} a = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right)$$

Projection matrix P

$$p = \hat{x} a = a \hat{x} = a \cdot \frac{a^T b}{a^T a}$$

$$= \left(\frac{a \cdot a^T}{a^T a}\right) \cdot b$$

Let $P = \dfrac{a \cdot a^T}{a^T a}$, then $p = Pb$

Ex 2. Find the projection matrix ~~th~~ onto the line through $a = (1, 2, 2)$.

$$P = \frac{a a^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

Let $b = (1, 1, 1)$, then

$$Pb = \frac{1}{9} \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} = p.$$

Note: 1° $P^2 = P$

(projecting a second time does not change anything)

2° $I-P$ is also a projection

When P projects onto one subspace, $I-P$ projects onto the perpendicular subspace

Projection Onto a Subspace

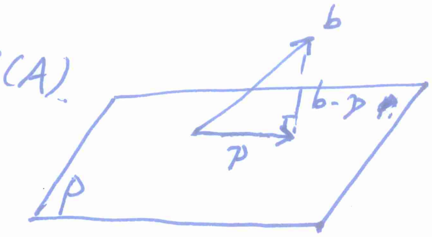
Let $a_1, \dots, a_n$ be vectors in $\mathbb{R}^m$, Assume a's are linearly independent

Question: Find $p = \hat{x}_1 a_1 + \dots + \hat{x}_n a_n$ closest to a given vector b.

Let $A = [a_1, \dots, a_n]$, $p = A\hat{x}$

$x = (\hat{x}_1, \dots, \hat{x}_n)$

Idea: $b-p$ is perpendicular to $C(A)$



$\Leftrightarrow$

$$a_1^T \cdot (b - A\hat{x}) = 0$$
$$\vdots$$
$$a_n^T (b - A\hat{x}) = 0$$

$$\Leftrightarrow \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} (b - A\hat{x}) = 0$$

$$\Leftrightarrow A^T(b - A\hat{x}) = 0$$

$$\Leftrightarrow A^TA\hat{x} = A^Tb$$

Since columns of A are independent, we can show that A^TA is invertible.

Thus, $\hat{x} = (A^TA)^{-1}A^Tb$

$$p = A\hat{x} = (A(A^TA)^{-1}A^T)b$$
$$\cancel{= [A(A^TA)^{-1}A^T]b}$$

Let $P = A(A^TA)^{-1}A^T$, then P is the projection matrix for $C(A)$.

Note: 2° $n=1$, $\hat{x} = \frac{a^Tb}{a^Ta}$ 1° $P^2 = P$

$p = a \cdot \frac{a^Tb}{a^Ta}$ $P^T = P$

$P = \frac{aa^T}{a^Ta}$.

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Ex 3. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

Find $\hat{x}$ ~~and~~, p, and P.

$$A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$A^TA\hat{x} = A^Tb \iff \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

$$P = A(A^TA)^{-1}A^T$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

Theorem A^TA is invertible $\iff$ A has (linearly) independent columns.

~~$\Leftarrow$ A has linear~~ Fact: A has independent columns $\iff$ $Ax=0$ has only zero solution

Pf: Idea: $N(A^TA) = N(A)$

$\supseteq$: clear.

$\subseteq$: Assume $A^TAx = 0$. ~~for some x~~

Then $x^TA^TAx = 0$

$$(x^TA^T)(Ax) = 0$$

$$(Ax)^T(Ax) = 0$$

$$\|Ax\|^2 = 0$$

$$\Rightarrow Ax = 0$$

(6) Note: When A has independent columns, A^TA is square, symmetric and invertible.