① Recall: Let $A$ be a $m \times n$ matrix with rank $r$

$$A \xrightarrow{\text{Gauss-Jordan}} R$$

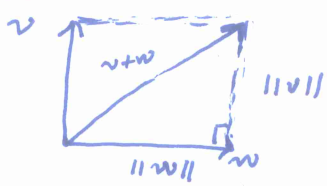
1. $C(A^T) = C(R^T)$ has dimension $r$ with pivots rows of $R$ as a basis
2. $C(A)$ has dimension $r$ with pivot columns of $A$ as a basis
3. The $n-r$ special solutions for $Rx=0$ are a basis for $N(A) = N(R)$
4. The $m-r$ special solutions for $A^Tx=0$ are a basis for $N(A^T)$.

# Chapter 4 Orthogonality

## §4.1 Orthogonality of the Four Subspaces.

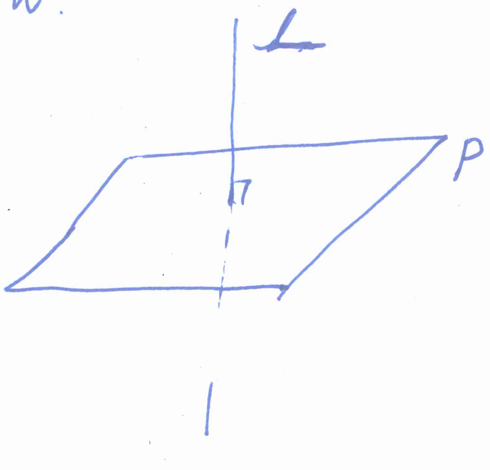
Orthogonal vectors $v \cdot w = v^T w = 0$

$\Rightarrow \|v\|^2 + \|w\|^2 = \|v+w\|^2$ Pythagoras Law




Def Let $V$ an $W$ be subspaces of a vector space.
$V$ and $W$ are orthogonal if $v^T w = 0$ for all $v$ in $V$ and all $w$ in $W$.

Ex.



② Let $A$ be a $m \times n$ matrix.

$N(A)$ and $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$

Consider $Ax = 0$

$$Ax = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} x = \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ each row vector of $A$ is perpendicular to $x$.

$\Rightarrow$ $C(A^T)$ is orthogonal to $N(A)$

Ex 3. (numerical example)

Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$Ax = 0 \Leftrightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 1+3-4=0 \\ 5+2-7=0 \end{cases}$$

Similarly, $N(A^T)$ and $C(A)$ are orthogonal in $\mathbb{R}^m$.

Orthogonal Complements.

~~Motivation~~ Fact: dimension of $N(A)$ + dimension of $C(A^T) = n$. < Let $A$ be a $m \times n$ matrix.

Def. The orthogonal complement of a subspace $V$ contains every vector that is perpendicular to $V$. This orthogonal space is denoted by $V^\perp$.

$$C(A^T)^\perp = N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$N(A)^\perp = C(A^T) \text{ (~~it can be shown~~) (by Fact)}$$

Similarly, $C(A)^\perp = N(A^T)$

$$N(A)^\perp = C(A)$$

} Fundamental Theorem of Linear Algebra, Part 2.

Note: the only vector in two orthogonal subspaces is the zero vector.

③ ~~Assume~~ If $v \cdot v = v^T \cdot v = 0$, then $v = 0$

Combining Bases from Subspaces (starting with correct number of vectors, one property of a basis produces the other one)

Any $n$ independent vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$. So they are a basis.

Any $n$ vectors that span $\mathbb{R}^n$ must be independent. ~~So they~~

$\Updownarrow$ Let $A = [v_1, \cdots, v_n]$ be a $n \times n$ matrix

1. If $n$ columns of $A$ are independent, they span $\mathbb{R}^n$. So $Ax = b$ is solvable
2. If $n$ columns span $\mathbb{R}^n$, they are independent. So, $Ax = b$ has only one solution.

(uniqueness implies existence and existence implies uniqueness).

Pf: $[A\ b] \xrightarrow{\text{Gauss-Jordan}} [R\ \ d]$

1. ~~R has n pivot columns, Rx = d is solvable~~ ~~$R = I$~~ $R$ has $n$ pivot columns $\Rightarrow R = I$ for any $b$
   $\Rightarrow Ax = b$ is solvable for any $b$
2. $Ax = b$ is solvable for any $b$
   $\Rightarrow R$ has $n$ pivot ~~columns~~ rows $\Rightarrow R = I$
   $\Rightarrow Ax = 0$ has only zero solution.

Let $v_1, \cdots, v_r$ be a basis for $C(A^T)$
$w_1, \cdots, w_{n-r}$ — — — $N(A)$

We can show that $v_1, \cdots, v_r, w_1, \cdots, w_{n-r}$ are independent
So they are a basis of $\mathbb{R}^n$

$\Rightarrow$ for each $x \in \mathbb{R}^n$, $x = x_r + x_n$, $x_r \in C(A^T)$, $x_n \in N(A)$.

Ex 5.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ split } x = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ into } x_r + x_n = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(I will show you how to make this splitting by a project in my next ~~lecture~~ lecture)