① Recall: 1. A basis consists of linearly independent vectors that span the space.

2. All bases for a space have the same number of vectors. The number is the dimension of the space.

§ 3.5 Dimensions of the Four Subspaces (related to a matrix)

Let $A$ be a $m \times n$ matrix with rank $r$.

| | | dimension |
|---|---|---|
| row space | $C(A^T) \subset \mathbb{R}^n$ | $r$ |
| null space | $N(A) \subset \mathbb{R}^n$ | $n-r$ |
| column space | $C(A) \subset \mathbb{R}^m$ | $r$ |
| left nullspace | $N(A^T) \subset \mathbb{R}^m$ | $m-r$ |

} Fundamental Theorem of Linear Algebra, Part 1 (relate rank and dimensions of subspaces of a matrix).

Note: $N(A^T) = \{ x \in \mathbb{R}^m \mid A^T x = 0 \}$

$= \{ x^T \in \mathbb{R}^m \mid x^T A = 0 \}$ - left nullspace

$$A \xrightarrow{\text{Gauss-Jordan}} R$$

The Four Subspaces for $R$

pivot columns

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \text{rows.} \end{array}$$

1. The row space of $R$ has dimension 2, matching the rank.

General case: The dimension of the row space is the rank $r$.

Nonzero rows of $R$ form a basis

2. The column space of $R$ has dimension $r=2$.

— — — : ~~The dimen~~ — — column — — —

pivot colums form a basis.

3. $R$ has 3 free variables $x_2, x_3, x_5$. Get

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_5 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$Rx = 0 \Rightarrow x = x_2 s_2 + x_3 s_3 + x_5 s_5$

$s_2, s_3, s_5$ are independent

The nullspace of $R$ has dimension $n - r = 5 - 2 = 3$.

② The nullspace has dimension $n-r$. The special solutions form a ~~b~~ basis.

4. Consider
$$y^T R = 0$$

$$\begin{array}{rl} & y_1 [1, 3, 5, 0, 7] \\ + & y_2 [0, 0, 0, 1, 2] \\ + & y_3 [0, 0, 0, 0, 0] \\ \hline & [0, 0, 0, 0, 0] \end{array}$$

$\Rightarrow y_1 = 0 = y_2$, $y_3$ is free

$N(R^T) = y_3 (0, 0, 1)$ has dimension $m-r = 3-2 = 1$

General case: $N(R^T)$ has dimension $m-r$, The solutions

$$(\underbrace{0, \cdots, 0}_{r}, 1, 0 \cdots, 0)$$
$$(0, \cdots, 0, 0, 1, \cdots, 0)$$
$$\vdots$$
$$(0, \cdots, 0, 0, 0, \cdots, 1)$$

form a basis.

The Four Subspaces for A

The subspace dimensions for A are the same as for R.

1. $C(A^T) = C(R^T)$. Same dimension $r$ and same basis.
2. ~~The column space of~~ $C(A)$ has dimension $r$. The column rank = the row rank. (Rank Theorem).

Note: 1. $C(A) \neq C(R)$

2. $Ax = 0 \Leftrightarrow Rx = 0$

dependent in $A \Leftrightarrow$ dependent in $R$

$\Rightarrow$ The $r$ pivot columns of $A$ are a basis for $C(A)$.

3. $N(A) = N(R)$. Same dimension $n-r$ and same basis

Counting Theorem: dimension of $C(A)$ + dimension of $N(A) = n$.

③ $N(A^T)$ has dimension $m-r$.

By counting rule, dimension of $N(A^T)$ + dimension of $C(A^T) = m$.

Fundamental Theorem of Linear Algebra, Part 1.

| | dimension |
|---|---|
| $C(A)$ | $r$ |
| $N(A)$ | $n-r$ |
| $C(A^T)$ | $r$ |
| $N(A^T)$ | $m-r$ |

Ex 1. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ has rank 1

$\qquad\qquad\quad\uparrow\ \ \uparrow$ free variables

$C(A^T)$ is a line in $\mathbb{R}^3$ spanned by $(1,2,3)$.

$N(A)$ has dimension 2.

$C(A) = \mathbb{R}^1$ with a basis 1

$N(A^T) = \{0\}$

$y_1 (1\ 2\ 3) = 0 \Rightarrow y_1 = 0$

Ex 2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ with rank $r = 1$. $\quad A \longrightarrow \begin{array}{ccc} & \text{free variables} & \\ & \downarrow & \downarrow \end{array}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R$

$C(A^T)$ is a line in $\mathbb{R}^3$ spanned by $(1,2,3)$.

$N(A)$ has dimension 2

$C(A)$ has dimension 1 and is spanned by $(1,2)$.

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \longrightarrow \begin{array}{cc} & \text{free variable} \\ & \downarrow \end{array}$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$N(A^T)$ has dimension 1, and has a basis $(2,-1)$.

Ex 3. Consider incidence matrix

$$A = \begin{bmatrix} -1 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 1 \end{bmatrix}$$

④ Consider $Ax = 0$

$\Rightarrow \cancel{x_1 = x_2 = x_3}$

$$\left\{\begin{aligned} x_1 &= x_2 \\ x_3 &= x_1 \\ x_2 &= x_4 \end{aligned}\right.$$

$\Rightarrow N(A) = c\cdot(1,1,1,1)$ has dimension 1.

$$A \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$C(A)$ has dimension 3 and is spanned by ~~row~~ column 1, 2, 3 of $A$.

$C(A^T)$ has a basis ~~of~~ $(1,0,0,-1), (0,1,0,-1), (0,0,1,-1)$.

$N(A^T)$ has dimension. $5-3=2$.

$A^T y = 0$ has ~~two~~ two solutions $(1,-1,1,0,0)$

$(0,0,-1,1,1)$.