①

Recall:

$Ax = b$

$[A\ b] \longrightarrow [R\ d]$

1° $Ax=b$ is solvable $\Leftrightarrow$ zero rows of R has zeros in d

2° If $Ax=b$ has a solution, one can get a ~~par~~ ~~particular~~ solution x_p ~~be~~by setting free variables to be zeros.

3° a complete solution of $Ax=b$ is

$$x = \underset{\substack{\uparrow \\ \text{particular}}}{x_p} + \underset{\substack{\uparrow \\ \text{nullspace}}}{x_n}$$

§ 3.4 Independence, Basis and Dimension

Motivation: What is the true size of a subspace (of $\mathbb{R}^3$)?

Linear independence < Let A be a matrix.

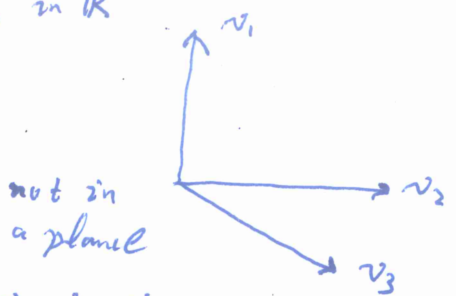
Def 1 columns of A are linearly independent $\Leftrightarrow$ $Ax=0$ has only zero solution. (No other combination Ax of the columns gives the zero vector)

No~~te: $A=[v_1, \dots, v_n]$, $x=(x_1, \dots, x_n)^T$~~

~~$Ax = x_1 v_1 + \dots + x_n v_n = 0$.~~

~~linearly independent $\Leftrightarrow$ if $x_1 v_1 + \dots + x_n v_n = 0$, then $x_1 =$~~

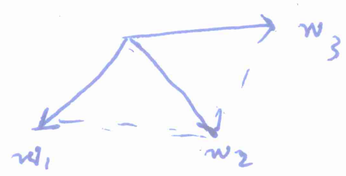
Ex: in $\mathbb{R}^3$



~~v_1, v_2, v_3 are not in the a same plane~~

$\Rightarrow$ if $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$, then $x_1 = x_2 = x_3 = 0$

②



$w_1 - w_2 + w_3 = 0$

Def 2. vectors $v_1, \dots, v_n$ is linearly independent $\Longleftrightarrow$ (vectors are not necessarily coordinate vectors ~~vectors in $\mathbb{R}^m$~~)

the only combination that gives zero vector is $0v_1 + \cdots + 0v_n$

Note: linear independence $\Longleftrightarrow$ if $x_1 v_1 + \cdots + x_n v_n = 0$, then

$$x_1 = x_2 = \cdots = x_n = 0$$

Ex: In $\mathbb{R}^2$:

(1) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are independent

(2) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.00001 \end{bmatrix}$ are independent.

(3) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ are dependent (take $x_1 = x_2 = 1$).

(4) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are dependent (take $x_1 = 0$, $x_2 = 1$)

(5) $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}$ are dependent since A is 2×3 matrix, $Ax = 0$ has nozero solution.

Ex 1. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

columns of A are dependent.

$$-3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(get it by solving $Ax = 0$ with elimination)

③ Note: ~~columns of full column rank matrix are d~~

1. columns of a matrix are independent $\Leftrightarrow$ A has full column rank ($N(A) = \{0\}$)

2. Any set of n vectors in $\mathbb{R}^m$ must be ~~independent~~ independent if $n > m$ (Fact: Let A be $m \times n$ with $n > m$, $Ax = 0$ has a nonzero solution)

Vectors that span a Subspace

Def 3 A set of vectors spans a ~~subspace~~ vector space if their linear combinations fill the space.

Ex: $C(A)$

Ex 2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$

Ex 3. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$

Ex 4. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ span a line in $\mathbb{R}^2$.

Def 4. Let A be $m \times n$ matrix
The row space of A is $C(A^T)$.

Note: $C(A^T)$ is a subspace of $\mathbb{R}^n$

Ex 5. ~~Describe C(A).~~
Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

Describe $C(A)$ and $C(A^T)$

$C(A)$ is a plance in $\mathbb{R}^3$ spanned by columns of A.

$C(A^T) = \mathbb{R}^2$ spanned by rows of A.

(A^T has full row rank)

① Recall:

1° $v_1, \dots, v_n$ are linearly independent $\Leftrightarrow$

if $x_1 v_1 + \cdots + x_n v_n = 0$, then $x_1 = x_2 = \cdots = x_n = 0$

2° $v_1, \dots, v_n$ span a space if their combinations fill that space.

A Basis for a Vector Space < Question: what are minimal spanning vectors for a vector space?

2 independent vectors can not span $\mathbb{R}^3$

4 vectors in $\mathbb{R}^3$ must be dependent (even if they span $\mathbb{R}^3$)

Want: enough independent vectors to span the space

(minimal spanning vectors for a space)

Def A basis for a vector space is a set of vectors with:

1° they are linearly independent;

2° they span the space.

Note: there is a unique way to write v as a combination of the basis vectors.

Pf: Suppose

$$v = a_1 v_1 + \cdots + a_n v_n \quad ①$$

$$v = b_1 v_1 + \cdots + b_n v_n \quad ②$$

① − ②

$$0 = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n$$

Since $v_1, \dots, v_n$ are independent, we have

$a_i = b_i$ for each i

Ex 6. columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce a basis for $\mathbb{R}^2$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — — $\mathbb{R}^3$.

② Note: ~~bases the basis~~ the bases are not unique!

Ex 7. columns of each invertible $n\times n$ matrix give a basis of $\mathbb{R}^n$?

$$A = \begin{bmatrix} 1 & 0 & \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Since $Ax=0$ has only zero solution, columns of A are independent

~~A also~~ Since $Ax=b$ is solvable for each $b\in\mathbb{R}^n$, $C(A)=\mathbb{R}^n$

Thus, columns of A are a basis of $\mathbb{R}^n$

$B=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ is singular $\Rightarrow$ columns of B are dependent

$\Rightarrow$ columns of B are not a basis of $\mathbb{R}^n$.

Note: 1° ~~vectors~~ $v_1, \cdots, v_n$ are a basis of $\mathbb{R}^n \Leftrightarrow$ they are columns of an $n\times n$ ~~matrix~~ invertible matrix.

2° $\mathbb{R}^n$ has infinitely many different bases.

Given a matrix A, how to compute a basis of $C(A)$?

Answer: pivot columns of A are a basis for $C(A)$ (independence, ~~span~~ fill the space)

pivot rows — A — — — for $C(A^T)$

$A \xrightarrow{\text{Gauss-Jordan}} R$

— rows — R — — — for $C(A^T) = C(R^T)$

Ex 8.

$$A = \begin{array}{c} \begin{array}{cc} v_1 & \end{array} \\ \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \end{array}$$

v_1 is the pivot column of A, also a basis of $C(A)$

$A \xrightarrow{\text{Gauss-Jordan}} R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$C(A^T) = C(R^T)$

$(1, 2)$ is a basis of $C(A^T)$.

③ Ex 9.

Let

$$R = \begin{array}{c} \text{pivot columns} \\ \begin{array}{cccc} \downarrow & & \downarrow & \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \leftarrow \\ \leftarrow \\ \end{array} \end{array}$$

Find bases for $C(R)$ and $C(R^T)$

$C(R)$ is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$C(R^T)$ —

$(1,2,0,3), (0,0,1,4)$

Question 1: Given 5 vectors in $\mathbb{R}^7$, how to find a basis for the space they span?

Answer: make them rows of A,

$$A \xrightarrow{\text{Gauss-Jordan}} R$$

nonzero rows of R are a basis for the space they span

Question 2: do bases of a space have the same number of vectors?

Dimension of a Vector Space

Theorem If $v_1, \dots, v_m$ and $w_1, \dots, w_n$ are both bases for the same vector space, then $m=n$.

Pf: Suppose $n > m$.

Since v's are a basis, $w_i = a_{1i} v_1 + a_{2i} v_2 + \dots + a_{mi} v_i$ $\quad 1 \le i \le n$.

Thus,

$$W = [w_1 \ w_2 \cdots w_n] = [v_1 \cdots v_m] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = VA$$

Since A is $m \times n$ matrix with $n > m$, $Ax = 0$ has a nonzero solution x

Thus, $Wx = VAx = 0$

$Wx = x_1 w_1 + x_2 w_2 + \dots + x_n w_n = 0$, $x_i \neq 0$ for some i

a contradiction with $w_1, \dots, w_n$ are independent.

④

1° The number of basis vectors depends only on the ~~basis~~ vector space

2° the ~~number~~ dimension counts the "degrees of freedom" in the space.

Def The dimension of a space is the number of vectors in every basis.

Ex: $\mathbb{R}^n$ has dimension n.

Ex: Let ~~N(A)~~ $A = [1 \ 5 \ 2]$. $N(A)$ has a basis $(-5, 1, 0)$ and $(-2, 0, 1)$. $N(A)$ has dimension 2.