1

Recall:

1. $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$ is a subspace of $\mathbb{R}^n$.

2. $A \xrightarrow{\text{Gauss-Jordan}} R$

   R contains pivot columns and free columns

3. Each free column leads to a special solution.

   free variable $= 1$, the others are 0

4. ~~Theo~~ rank ~~of~~ r of A is the number of pivots

5. $N(A) =$ all combinations of $n-r$ special solutions

6. Let A be $m \times n$ matrix with $n > m$. Then $Ax = 0$ has a nonzero solution

§3.3 The complete Solution to $Ax=b$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

Consider (augmented matrix)

$$\underset{[A\ b]}{\underset{\parallel}{\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right]}} \xrightarrow{\text{Gauss-Jordan}} \underset{[R\ d]}{\underset{\parallel}{\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}}$$

Thm $Ax=b$ has a solution ~~because~~ $\overset{?}{\Leftarrow} \Leftrightarrow \underset{?}{\Rightarrow}$ zero rows of R has zeros in d

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = [R\ b]$$

$Ax = b \Leftrightarrow b_3 = b_1 + b_2$

② One Particular Solution $Ax_p = b$

$[A \ b] \longrightarrow [R \ d]$

if $Ax = b$ has a solution, then zero rows in R must be zeros in d

Choose free variables $= 0$, then pivot variables come from d.

$$[R \ d] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = x_4 = 0$, get $x_1 = 1$, $x_3 = 6$

$x_p = (1, 0, 6, 0)$

Let A be $m \times n$ matrix with rank r

$x_p =$ ~~x particular~~ particular solution solves $Ax_p = b$

x_n $\quad$ $n-r$ special solution solve $Ax_n = 0$

$x = x_p + x_n$ $\quad$ complete solution to $Ax = b$

$Rx = 0$ has two solutions

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$Ax = b \Rightarrow x = x_p + x_n$

$$= \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \quad x_2, x_4 \text{ be any number}$$

Question: ~~If~~ Let A be a square invertible matrix. What are x_p and x_n?

$x_p = A^{-1} b$

$x_n = 0$

$x = x_p + x_n = A^{-1} b$

③ Ex 1.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

· When does $Ax = b$ has a solution?

$$[A \mid b]$$

$$\parallel$$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{array}\right] = [R \; d]$$

$Ax = b$ is solvable $\iff b_1 + b_2 + b_3 = 0$

Assume $b_1 + b_2 + b_3 = 0$ > Since $n - r = 0$, $Ax_n = 0 \Rightarrow x_n = 0$

$Ax_p = b \Rightarrow x_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$

$Ax = b \Rightarrow x = x_p + x_n = x_p$

If A has full column rank ($r = n$), then

$$A \longrightarrow R = \begin{array}{c} n \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \end{array} n$$

$N(A) = \{0\}$.

In this case, $Ax = b$ has only one solution or no solution.

The Complete Solution.

Let A be $m \times n$ matrix with rank r.

A has full row rank if $r = m$.

Ex 2 Consider

$$\begin{cases} x + y + z = 3 \\ x + 2y - z = 4 \end{cases}$$

(4) Solutions form a line in $\mathbb{R}^3$

$$[A \ b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [R \ d]$$

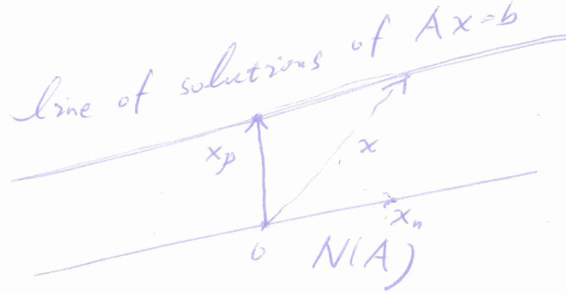
no zero rows in $R \Rightarrow Ax = b$ has a solution

Set $x_3 = 3$, get $x_p = (2, 1, 0)^T$

$Rx = 0$

Set $x_3 = 1$, get $s = (-3, 2, 1)$

complete solution of $Ax = b$ is



$$x = x_p + x_n$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \quad x_3 \text{ is any number}$$

If A has full row rank ($r = m$), then

1. R has no zero rows
2. $Ax = b$ has a solution for any b
3. $C(A) = \mathbb{R}^m$ (take $b = e_1, \dots, e_m$)

(4)

4 possibilities for linear equations

	R	$Ax = b$
$r = m, \ r = n$	$[I]$	1 solution
$r = m, \ r < n$	$[I \ F]$	∞ ...
$r < m, \ r = n$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	0 or 1 ...
$r < m, \ r < n$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	0 or ∞ ...