①

Recall: 1. all combinations of columns of A (A $m \times n$ matrix) form the column space $C(A)$. $C(A)$ is a subspace of $\mathbb{R}^m$

2. $Ax=b$ has a solution $\Leftrightarrow$ $b \in C(A)$

§ 3.2 Nullspace of A: Solving $Ax=0$ and $Rx=0$

Let A be $m \times n$ matrix

Consider

$$Ax=0$$

- $x=0$ is a solution

if $m=n$ and A is invertible, $Ax=0 \Rightarrow x=0$

(We are considering the general case)

Def: The nullspace $N(A) = \{ x \in \mathbb{R}^n \mid Ax=0 \}$

$N(A)$ is a subspace of $\mathbb{R}^n$

Pf: Let $x, y \in N(A)$, $c \in \mathbb{R}$.

1° $A(x+y) = Ax + Ay = 0+0=0 \Rightarrow x+y \in N(A)$

2° $A(cx) = (Ac)x$
$= (cA)x$
$= c(Ax)$
$= c \cdot 0$
$= 0$

$\Rightarrow cx \in N(A)$

Ex 1.
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Describe $N(A)$.

$$Ax=0 \Leftrightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases} \longrightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 0 = 0 \end{cases}$$

$N(A)$ is a line through $(0,0)$ in $\mathbb{R}^2$

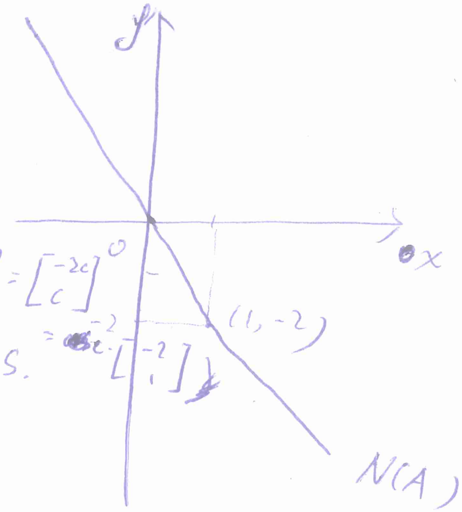
$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2\underset{\substack{\uparrow \\ \text{free variable}}}{x_2}$

② Set $x_2 = 1$, then $x_1 = -2$.

We get a (special) solution $s = (-2, 1)$

(Once we give values to x_2, we can determin x_1) ~~$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$~~ Let $x_2 = c$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2c \\ c \end{bmatrix}$

$N(A)$ ~~contains~~ $=$ all multiples of s. $= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



~~Ex 2~~ Note: $N(A)$ con~~sists of~~ $=$ all combinations of special solutions of $Ax = 0$

Ex 2. $A = [1 \ \ 2 \ \ 3]$

$Ax = 0 \iff x + 2y + 3z = 0$ (a plane in $\mathbb{R}^3$)

$\iff x = -2y - 3z$, ~~y, z are free variables~~

y, z ↑ free variables

Set $y = 1$, $z = 0$, $\Rightarrow x = -2$

$y = 0$, $z = 1$, $\Rightarrow x = -3$

let $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Then s_1, s_2 are special sols of $Ax = 0$

Set $y = c_1$, $z = c_2$, c_1, c_2 are (arbitrary) numbers

$x = -2c_1 - 3c_2$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2c_1 - 3c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{= s_1} + c_2 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{= s_2}$$

$N(A)$ con~~sists~~ $=$ all combinations of s_1 and s_2.

(What about the general case?)

Pivot Columns and Free Columns

$A = [1 \ \ 2 \ \ 3]$

↑ pivot column; ↑↑ free columns

(The special choice is only for free variables. The pivot variables are uniquely determined by

③

Ex 3.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad C = [A \;\; 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

$$Ax = 0 \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\uparrow \uparrow$ pivot column — no free variables

$$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \xrightarrow{r_2 - 3r_1} U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$\uparrow \uparrow$ pivot column $\quad \uparrow \uparrow$ free columns

Set $x_3 = 1$, $x_4 = 0$, get $s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (x_1, x_2 are pivot variables)
(free variables)

Set $x_3 = 0$, $x_4 = 1$, get $s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

$N(C)$ = con~~sists of~~ all combinations of s_1, s_2.

Note: elimination does not change solutions.

The Reduced Row Echelon Form R (tells us the structure of $N(A)$)

$$A \xrightarrow{\text{elimination}} U \xrightarrow{\text{1 and 2}} R$$

1. Produce zeros above the pivots
2. Produce ones in the pivots.

R is called the reduced row echelon form of A, denoted by rref(A)

Note: 1. $N(A) = N(U) = N(R)$

2. The pivots columns of R contain I.

④

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \end{array} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{array}{l} \\ r_2 \end{array} \xrightarrow{\frac{1}{2} r_2} \overset{R}{\overset{\parallel}{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}}}$$

piv~~free~~ columns free columns

Pivot Variables and Free Variables in R

$$A \longrightarrow R = \begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 Pivot variables $x_1 \quad x_2 \quad x_3$

2 free variables $x_3 \quad x_5$

set $x_3 = 1$, $x_5 = 0$, get $s_1 = (-a, -b, 1, 0, 0)^T$

$x_3 = 0$, $x_5 = 1$, get $s_2 = (-c, -d, 0, -e, 1)^T$

$N(A)$ = all combinations of s_1 and s_2.

Let A be $m \times n$ matrix. $Ax = 0$ < variables { free variables / pivot

If $n > m$, then ~~number of pivots ≤~~ number of pivots $\le m < n$

$\Rightarrow$ there ~~are~~ is at least one free ~~number~~ variable.

Theorem Let A be $m \times n$ matrix with $n > m$. Then $Ax = 0$ has nonzero solutions

The Rank of a Matrix.

$Ax = 0$ what is the true size of A? ($N(A) = N(R)$) (the numbers of columns and rows of A does not reflect true size of A)

Def The rank of A is the number of pivots.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank of A is 2

⑤ Rank one

1. Matrices of rank one have only one pivot.
2. Every row is a multiple of the pivot row

$$A = \begin{bmatrix} 1 & \cancel{2}3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 10 \end{bmatrix} = u v^T$$

$$Ax = 0 \Leftrightarrow u v^T x = 0 \Leftrightarrow v^T x = 0$$