1

Recall:

1. $(A^T)_{ij} = A_{ji}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

2. $(AB)^T = B^T A^T$

   $(A^{-1})^T = (A^T)^{-1}$

3. dot product: $x \cdot y = x^T y$

   $(Ax)^T y = x^T (A^T y)$

4. If $S = S^T$ (symmetric), $S = LDL^T$

5. A permutation matrix P has rows of I in any order

   $P^{-1} = P^T$

6. If A is invertible, $PA = LU$

Chapter 3.

§3.1 Spaces of Vectors

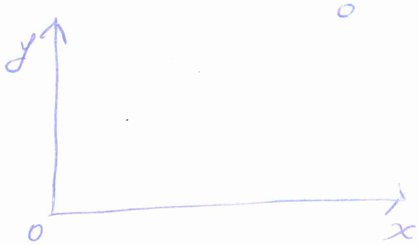
Let $\mathbb{R}$ be ~~real~~ real numbers.

$\mathbb{R}^1 = \{0, 1, -1, \pi \cdots\}$

(a line)

$\mathbb{R}^2$

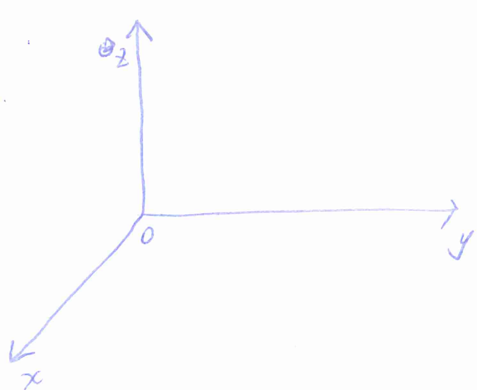
$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2$



(xy plane)

$\mathbb{R}^3$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$



(3-D space)

②

Def $\mathbb{R}^n$ consists of all collumn vectors v with n components

$(1, 1, 0, 1, 1) \in \mathbb{R}^5$

~~In $\mathbb{R}^n$, we can we can add two vectors, and we can multiply~~

Let $u, v \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then

1° $cv \in \mathbb{R}^n$ (closed under scalar multiplication)

2° $u+v \in \mathbb{R}^n$ (closed under addition.)

3° ~~$0 \in \mathbb{R}^n$~~ $0+v=v$

4° $v+u=u+w$ (commutative ~~law~~ law for addition)

5° $c(v+w)=cv+cw$ (distributive law for addition)

there are 8 laws for each vector space (see problem set)

A (real) vector space is a set of "vectors" together with rules for vector addition and scalar multiplication

Ex: $\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \in \mathbb{C}^2$

M The vector space of all real 2x2 matrices.

F ———————— real functions $f(x)$

Z ———————— only a zero vector.

Subspaces.

$\mathbb{R}^3$

a plance through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$

(It is a vector space in its own right. closed under addition and scalar multiplication).

③ Def A subspace of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies:

If v ~~and~~, w are in the subspace, c be any numbers. then

(i) $v+w$ is the subspace;

(ii) cv - - - ———

(all linear combination stay in the subspace)

subspace is a vector space (8 laws are automatically satisfied)

Fact: each subspace contains the zero vector

Take $c=0$, $c \cdot v = 0 \cdot v = 0$

Subspaces of $\mathbb{R}^3$:

(L) Any line through $(0,0,0)$

(P) - - - plane - - - ———

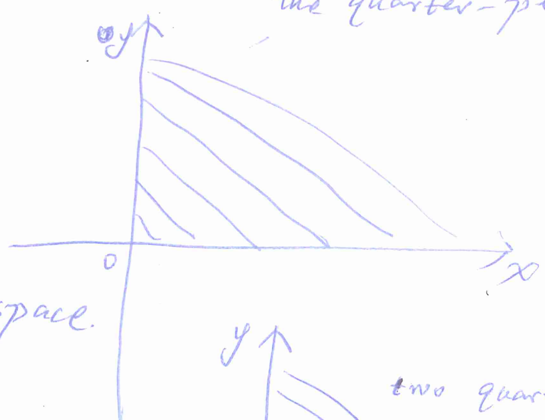
($\mathbb{R}^3$) the whole space

(Z) $(0,0,0)$

(~~non-subsp~~

(Examples for non-subspaces)

Ex 1: Consider the quarter-plane P_1.

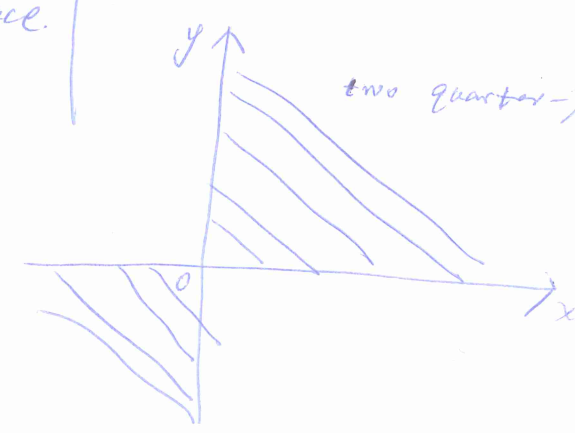


$(2,3) \in P_1$

$(-2,-3) \notin P_1$

P_1 is not a ~~subsp~~ subspace.

Ex 2. two quarter-plane P_2



$v = (2,3) \in P_2$

$w = (-3,-2) \in P_2$

$v+w = (-1,1) \notin P_2$

P_2 is not a subspace.

4

~~A subspace contains~~

Let V be a subspace, If $v, w \in V$, then $cv + dw \in V$ for any number c, d.

Ex 3. Let M be all 2×2 matrices

Consider

(U) all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(D) ——— $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

U, D is a subspace of M

D is a subspace of U.

① Recall:

1. a vector space has two operations:

   ① vector addition;

   ② scalar multiplication.

   ① and ② satisfy 8 laws.

   Ex: $\mathbb{R}^n$

2. A subspace containing v and w contain all combinations $cv+dw$.

The Column Space of A (the most important subspaces is related to a matrix A)

Let A be $m\times n$ matrix.

$A=[v_1, \dots, v_n]$

Ax is a combination of $v_1, \dots, v_n$.

(If we take all linear combinations of $v_1, \dots, v_n$. This produces the column space of A: It is a vector space made up of column vectors.)

Def The column space of A consists of all combination of $v_1, \dots, v_n$. We denote it by $C(A)$.

Note: We also say $C(A)$ is spanned by $v_1, \dots, v_n$.

Consider

$$Ax=b$$

$Ax=b$ is solvable if and only if $b\in C(A)$

Since $\underset{[v_1 \cdots v_n]}{A}$ is $m\times n$ matrix, $C(A)$ is a subspace of $\mathbb{R}^m$

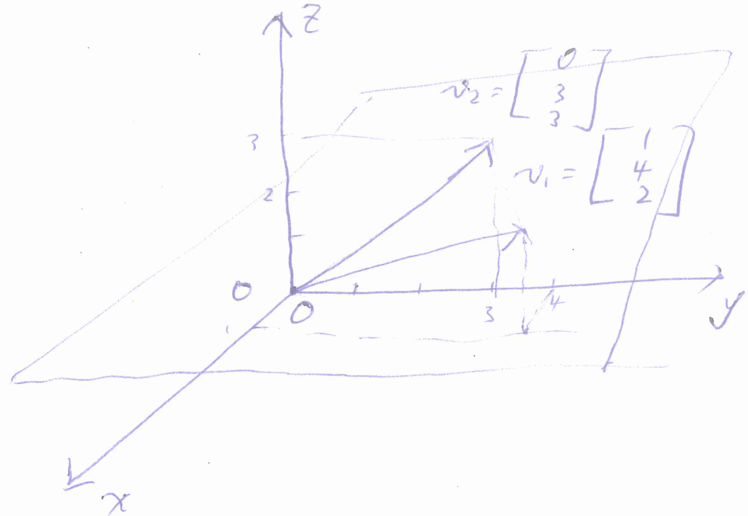
②

Ex 4.

$$A = \begin{array}{cc} v_1 & v_2 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$C(A)$ consists of all combinations of v_1, v_2.

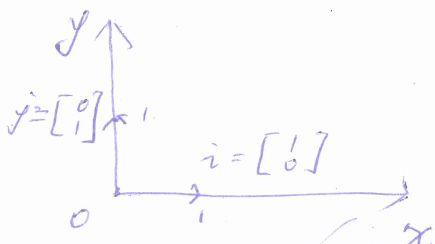
$C(A)$ is a plane in $\mathbb{R}^3$ through the origin.



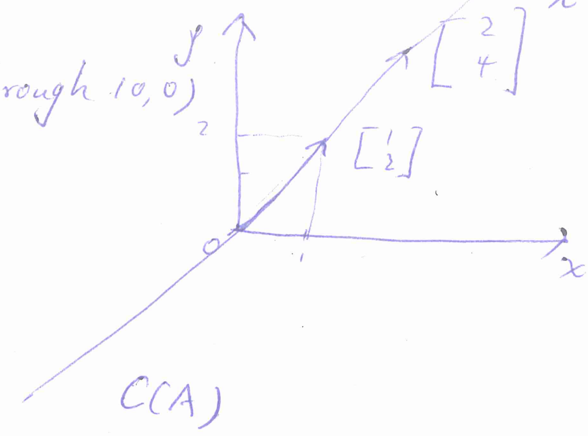
Ex 5.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$C(I) = \mathbb{R}^2$



$C(A)$ is a line through $(0,0)$



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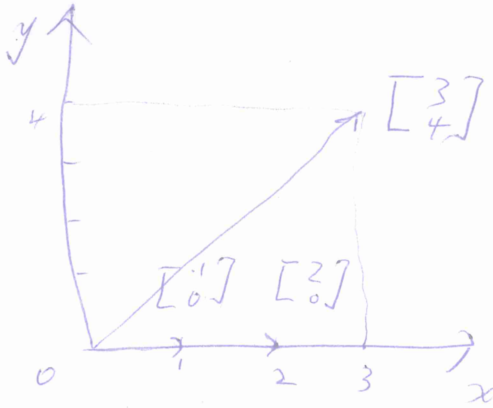
$C(B) = \mathbb{R}^2$



Review: Let A be $m \times n$

1. The combinations of colums of A form the column space $C(A)$. $C(A)$ is a subspace of $\mathbb{R}^m$ and spanned by colums.
2. $Ax = b$ has a solution $\Leftrightarrow$ $b \in C(A)$.