Recall:

1. elimination gives $A = L \cdot U$

2. Solve: $Ax = b$

   Factor: $A = LU$

   Solve: $Lc = b$, $Ux = c$.

§2.7 Transpose and Permutations.

Ex:

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \end{array} \quad \text{then } A^T = \begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix} \end{array}$$

Def

$$(A^T)_{ij} = A_{ji}$$

Sum: $(A+B)^T = A^T + B^T$

Product: $(AB)^T = B^T A^T$ (1)

Inverse: $(A^{-1})^T = (A^T)^{-1}$ (2)

Proof of (1):

Consider $(Ax)^T$

Ax is a combination of columns of A

$x^T A^T$ is a combination of rows of A^T

$\Rightarrow (Ax)^T = x^T A^T$

$B = [x_1, x_2 \cdots x_n]$

$AB = [Ax_1 \ Ax_2 \cdots Ax_n]$

$$(AB)^T = \begin{bmatrix} (Ax_1)^T \\ (Ax_2)^T \\ \vdots \\ (Ax_n)^T \end{bmatrix} = \begin{bmatrix} x_1^T A^T \\ x_2^T A^T \\ \vdots \\ x_n^T A^T \end{bmatrix} = B^T A^T.$$

Ex:

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 9 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 0 & 1 \end{bmatrix}$$

If $A = LDU$, then $A^T = U^T D^T L^T$

$$D = D^T$$

Proof of (2): $A^{-1}A = I$

$$(A^{-1}A)^T = I^T$$

$$A^T \cdot (A^{-1})^T = I$$

Ex 1: $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ with $A^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

$$(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}.$$

The Meaning of ~~Inner~~ dot Products.

1×1 $\quad x \cdot y = x^T \cdot y$ $\quad$ dot product (inner product)

$n\times n$ $\quad x \cdot y^T$ $\quad$ outer product

Motivation for transposes:

$$(Ax)^T y = x^T (A^T y)$$

Ex: $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{aligned}(Ax)^T \cdot y &= (x_2 - x_1) y_1 + (x_3 - x_2) y_2 \\ &= x_1(-y_1) + x_2(y_1 - y_2) + x_3(y_2) \\ &= x^T (A^T y)\end{aligned}$$

Symmetric Matrices.

3

Def A symmetric matrix $S^T = S$

Ex: $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = S^T$ (symmetric along the diagonal)

$D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^T$

Note: If S is symmetric, ~~then S^{-1} is~~ so is S^{-1}

$$(S^{-1})^T = (S^T)^{-1} = S^{-1}$$

Ex: $S^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$

Symmetric Products A^TA, AA^T and LDL^T (one way to product symmetric matrices)

Let A be $m \times n$.

Then $(A^TA)^T = A^T(A^T)^T = A^TA$

AA^T is also symmetric.

Ex 2. $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Symmetric matrices in elimination

~~Assume $S = S^T$~~

~~$S = LDU = LDL^T$ $(U = L^T)$, proof using lower triangular = upper triangular~~

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

④ If $S = S^T$ and $S = LDL^T$, then $U = L^T$

$$S = LDL^T$$

(This can be used to reduce computation for factorization)

Permutation Matrices

row exchange matrix P_{ij}

Def A permutation matrix P has the rows of ~~the~~ I in any order

Ex 3.

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

There are $\underset{\substack{\uparrow \\ n \text{ factorial}}}{n!} = (1)(2)\cdots(n)$ permutation matrices of order n.

Note ① P^{-1} is also a permutation matrix. (~~undo~~ undo ~~to~~ the permutation of P)

② $P^{-1} = P^T$

Pf: $PP^T = I$.

The $PA = LU$ with Row Exchanges

without ~~Row~~ row exchange

$$A = (E_{21}^{-1} \cdots E_{ij}^{-1} \cdots)U$$

with row exchange

$$A = (E^{-1} \cdots P^{-1} \cdots E^{-1} \cdots P^{-1} \cdots)U$$

⑤ Put all P_{ij}'s into a single permutation P. Then

$$PA = LU$$

$$\begin{array}{c} r_1 \\ r_2 \\ \\ \end{array}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}\begin{array}{c} r_1^* \\ \\ r_3^* \end{array} \xrightarrow{r_3^* - 2r_1^*} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}\begin{array}{c} \\ \bar{r}_2 \\ \bar{r}_3 \end{array}$$

$A \qquad\qquad PA$

$$\xrightarrow{\bar{r}_3 - 3\bar{r}_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$$