MATH 2418: Linear Algebra

Assignment 9 (sections 4.1, 4.2, 4.3 and 4.4)

Due: `April 10, 2019` Term: `Spring, 2019`

[First Name] [Last Name] [Net ID]

Suggested problems (do not turn in):Section 4.1: 1, 2, 3, 5, 6, 7, 10, 11, 14, 16 20, 21, 22, 24, 25, 26, 28 29; Section 4.2: 1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 16, 17, 21, 23, 24; Section 4.3: 1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 14, 17, 18, 19, 21, 22; Section 4.4: 1, 3, 4, 5, 6, 9, 13, 15, 18, 19, 22, 24. Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points] Find a vector orthogonal to the null space of matrix A, where

   (a) (5 pts) $A = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} -2 & 3 & -3 \end{bmatrix}$.

   (b) (5 pts) $A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ -3 & -6 & -9 & -12 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

Solution:

(a) $A = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -3 \\ -4 & 6 & -6 \\ 6 & -9 & -9 \\ -8 & 12 & -12 \end{bmatrix}$

The null space is $N(A) = span\left\{ (3,2,0), (-3,0,2) \right\}$.

Choose an orthogonal vector to be $v = (\frac{1}{3}, -\frac{1}{2}, \frac{1}{2})$. Indeed:

$\begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = 0,$

(b) The null space is $N(A) = span\left\{ (-2,1,0,0), (-3,0,1,0), (-4,0,0,1) \right\}$.

Again, by vector product, it is easy to see that vector orthogonal to $N(A)$ is $v = (1,2,3,4)$.

2. [10 points] Find a basis for the orthogonal complement to the row space of the matrix

$$\begin{bmatrix} 2 & -1 & 3 & 4 & -5 & 6 \\ 6 & -3 & -8 & 12 & -15 & 18 \\ 4 & -2 & 0 & 8 & -10 & 6 \\ 4 & -2 & 0 & 8 & -10 & 12 \end{bmatrix}$$

Solution:

Let,

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 & -5 & 6 \\ 6 & -3 & -8 & 12 & -15 & 18 \\ 4 & -2 & 0 & 8 & -10 & 6 \\ 4 & -2 & 0 & 8 & -10 & 12 \end{bmatrix}$$

Since the row space and null space of the matrix A are orthogonal complement to each other. So we need to find the basis for null space of A. i.e we need to find solution of $Ax = 0$.

By performing row operations, we get the reduced form of A:

$$\begin{bmatrix} 2 & -1 & 3 & 4 & -5 & 6 \\ 0 & 0 & 2 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

There are three free variables x_2, x_4, x_5, one basis for the orthogonal complement to the row space of the matrix A is $\{(1/2, 1, 0, 0, 0, 0), (-2, 0, 0, 1, 0, 0), (5/2, 0, 0, 0, 1, 0)\}$.

3. [10 points] Let P be the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation $-2x + y - 3z = 0$. Find a unit vector $\mathbf{n}$ orthogonal to P.

Solution:

The corresponding matrix is: $A = \begin{bmatrix} -2 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The null space is the set of points P, and the orthogonal component is row space of A, which is $R(A) = t(-2, 1, -3), t \in \mathbb{R}$.

A unit vector $\mathbf{n}$ in $R(A)$ is $\dfrac{(-2, 1, -3)}{\sqrt{(-2)^2 + 1^2 + (-3)^2}} = \left(\dfrac{-\sqrt{14}}{7}, \dfrac{\sqrt{14}}{14}, \dfrac{-3\sqrt{14}}{14} \right)$

4. [10 points] For the projection onto the vector $\mathbf{a} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

(a) (3 points) Find the projection matrix P.

Solution:

$$P = \frac{\boldsymbol{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \end{bmatrix}}{\begin{bmatrix} -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}}$$

$$\frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}$$

(b) (3 points) Project the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Solution:

$$P\mathbf{b}_1 = \frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P\mathbf{b}_2 = \frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

(c) (1 point) Find the errors $\mathbf{e}_1 = \mathbf{b}_1 - P\mathbf{b}_1$ and $\mathbf{e}_2 = \mathbf{b}_2 - P\mathbf{b}_2$

Solution:

$$\mathbf{e}_1 = \mathbf{b}_1 - P\mathbf{b}_1 = \mathbf{b}_1$$

$$\mathbf{e}_2 = \mathbf{b}_2 - P\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 31 \\ 13 \\ 40 \end{bmatrix}$$

5. [10 points] Find the minimal distance from the point $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 5 \end{bmatrix}$ to the space of all linear combinations of the vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

Solution:

Form the linear system $A\mathbf{x} = \mathbf{b}$
where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 5 \end{bmatrix}$$

To find the minimal distance, we need to solve the normal equation $A^T A\hat{x} = A^T\mathbf{b}$.

$$A^T A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

Solution of the normal equation:

$$\hat{x} = (A^T A)^{-1} A^T\mathbf{b} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$A\hat{x} - b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ -1 \\ -1 \end{bmatrix}$$

Therefore the required minimum distance:

$$\sqrt{(\,\|A\hat{x} - b\|)^2} = \sqrt{(-1)^2 + (-7)^2 + (-1)^2 + (-1)^2} = \sqrt{52}$$

6. [10 points] Find the best line $q(t) = At+B$ approximating the data set b at the times $t = 0, 1, 2, 3, 4$.

   (a) (5 points) $b = -1, -1, 2, 0, 0$

   (b) (5 points) $b = -1, 0, 2, 0, 1$

Solution:

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$$

and b be the given vector of values in the data set corresponding to times t. In order to find the best line approximating the given data set, one needs to minimize $\|Mx - b\|^2$ where $x = [A\ B]^T$. The latter is equivalent to solving the *normal* equation:

$$M^T M x = M^T b.$$

Computing $M^T M$ yields

$$M^T M = \begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix}.$$

In part (a), vector $b = [-1\ -1\ 2\ 0\ 0]^T$. Hence, $M^T b = [3\ 0]^T$, and normal equation is

$$\begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

The solution to the equation above is $A = 3/10$ and $B = -3/5$. Therefore, the answer is

$$q(t) = \frac{3}{10}t - \frac{3}{5}$$

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Similarly, in part (b), vector $b = [-1\ 0\ 2\ 0\ 1]^T$. Hence, $M^T b = [8\ 2]^T$, and normal equation is

$$\begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

The solution to the equation above is $A = 2/5$ and $B = -2/5$. Therefore, the answer is

$$q(t) = \frac{2}{5}t - \frac{2}{5}$$

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7. [10 points] Find the closest parabola $q(t) = At^2 + Bt + C$ approximating the data set

time t	-1	0	1	2	3
data q	5	2	1	2	5

Solution:

Form the linear system $M\mathbf{x} = \mathbf{q}$ where

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

To find the closest parabola, we need to solve the normal equation $M^T M\mathbf{x} = M^T\mathbf{q}$.

$$M^T M = \begin{bmatrix} 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 99 & 35 & 15 \\ 35 & 15 & 5 \\ 15 & 5 & 5 \end{bmatrix}$$

$$M^T\mathbf{q} = \begin{bmatrix} 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 59 \\ 15 \\ 15 \end{bmatrix}$$

Do Gaussian elimination to the augmented matrix:

$$\left[\begin{array}{ccc|c} 99 & 35 & 15 & 59 \\ 35 & 15 & 5 & 15 \\ 15 & 5 & 5 & 15 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 15 & 5 & 5 & 15 \\ 35 & 15 & 5 & 15 \\ 99 & 35 & 15 & 59 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 15 & 5 & 5 & 15 \\ 0 & \frac{10}{3} & -\frac{20}{3} & -20 \\ 0 & 2 & -18 & -40 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 15 & 5 & 5 & 15 \\ 0 & \frac{10}{3} & -\frac{20}{3} & -20 \\ 0 & 0 & -14 & -28 \end{array}\right]$$

So the solution is

$$\mathbf{x} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

and this gives us the equation of the closest parabola

$$q(t) = t^2 - 2t + 2$$

8. [10 points] Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ -3/5 \\ 4/5 \end{bmatrix}$.

(a) (5 points) Find vector $\mathbf{e}_3$ such that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal basis in $\mathbb{R}^3$.

Solution:
Let $e_3 = (k_1, k_2, k_3)$, then we have that $e_1 \cdot e_3 = 0$ and $e_2 \cdot e_3 = 0$, so:

$$e_1 \cdot e_3 = k_1 + 0 + 0 = k_1 = 0$$

$$e_2 \cdot e_3 = -\frac{3}{5}k_2 + \frac{4}{5}k_3 = 0 \Rightarrow k_2 = \frac{4}{3}k_3$$

But since we have that e_3 is a unit vector, we have that:

$$||e_3|| = 1 \Rightarrow \frac{16}{9}k_3^2 + k_3^2 = 1$$

$$\Rightarrow \frac{25}{9}k_3^2 = 1 \Rightarrow k_3^2 = \frac{9}{25}$$

Thus we have two solutions: $k_3 = \pm\frac{3}{5}$, thus taking the positive solution, we get the vector:

$$e_3 = \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix}$$

Clearly e_1, e_2, and e_3 are orthonormal.

(b) (1 point)In how many different ways can you choose $\mathbf{e}_3$?

Solution:
Note that we could have also chosen $k_3 = -\frac{3}{5}$, which would have given us the other solution:

$$e_3 = \begin{bmatrix} 0 \\ -4/5 \\ -3/5 \end{bmatrix}$$

So there are only two ways of choosing e_3.

(c) (4 points) Express vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Solution:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -3/5 \\ 4/5 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix}$$

$$\Rightarrow c_1 = 1, c_2 = 1/5, c_3 = 7/5$$

9. [10 points] Find orthogonal vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ by Gram-Schmidt from the vectors $\mathbf{a} = (1, -2, 2, 0)$, $\mathbf{b} = (-2, 2, 0, 1)$ and $\mathbf{c} = (-2, 0, 1, 2)$.

Solution:

Note first that $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$. Thus we may take $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$. To obtain $\mathbf{C}$, we let $\mathbf{C} = \mathbf{c} - proj_{\mathbf{c}}(\mathbf{a}) - proj_{\mathbf{c}}(\mathbf{b})$, where

$$proj_{\mathbf{v}}(\mathbf{u}) := \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||^2}\mathbf{u}$$

So $\mathbf{C} = \frac{1}{3}(-2, -4, 3, 4)$

10. [10 points] True or False? Circle your answer and **provide a justification** for your choice.

(a) **T** **F:** Row space and column space of the square matrix coincide.

Solution:
Consider the counterexample $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, it's readily seen $C(A) \neq R(A)$

(b) **T** **F:** If vector $\mathbf{a}$ belongs to the null-space of some matrix A then any $\mathbf{b}$ orthogonal to $\mathbf{a}$ belongs to the row space of A.

Solution:
Consider the counterexample $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, it's readily seen $R(A)$ is spanned by $[1\ 0\ 0]$, $\mathbf{a} = [0\ 1\ 0]^T$ belongs to $N(A)$, $\mathbf{b} = [1\ 0\ 1]$ is orthogonal to $\mathbf{a}$, but $\mathbf{b} \notin R(A)$

(c) **T** **F:** If vectors $\mathbf{b}$ and $\mathbf{a}$ are orthogonal then projection of $\mathbf{b}$ onto line through $\mathbf{a}$ has no errors.

Solution:
Taking any $\mathbf{b} \neq \mathbf{0}$ and a line $\boldsymbol{l}$ passing through $\mathbf{0}$ and $\mathbf{a}$, the projection of $\mathbf{b}$ onto $\boldsymbol{l}$ is $\mathbf{0}$ and the error is $\mathbf{b} \neq \mathbf{0}$

(d) **T** **F:** If matrix A is a square matrix then $A(A^TA)^{-1}A^T = I$.

Solution:
Taking any size n square matrix A such that $\text{rank}(A) < n$, since $\text{rank}(A^TA) = \text{rank}(A)$, A^TA will be singular and there is no inverse $(A^TA)^{-1}$.

(e) **T** **F:** Least square approximation finds the line passing through all points in the data set.

Solution:
If the least square aproximation is supposed to fit in a line, for example in a affine function, and the given points are not on the same line, then the least square aproximation can not pass through all points.

(f) **T** **F:** If P is a projection matrix then $P^3 = P$.

Solution:
If P is a projection matrix, then $P^2 = P$, therefore $P^3 = PP^2 = PP = P^2 = P$.

(g) **T** **F:** Projection of the vector onto the subspace minimizes the length of the error vector.

Solution:
Let $\mathbf{p}$ be the (orthogonal) projection of a vector $\mathbf{b}$ onto a subspace S. Setting $\hat{\mathbf{e}} = \mathbf{b} - \mathbf{p}$, by condition we have $\hat{\mathbf{e}}$ orthogonal to any vector on S, therefore given any $\mathbf{v} \in S$, we have $\mathbf{p} - \mathbf{v} \in S$ and $\|\mathbf{b} - \mathbf{v}\|^2 = \|\hat{\mathbf{e}} + \mathbf{p} - \mathbf{v}\|^2 = \|\hat{\mathbf{e}}\|^2 + \|\mathbf{p} - \mathbf{v}\|^2 \geq \|\hat{\mathbf{e}}\|^2$, hence $\|\mathbf{b} - \mathbf{v}\| \geq \|\hat{\mathbf{e}}\|$ for any $\mathbf{v} \in S$, therefore the projetion vector $\mathbf{p}$ minimizes the length of the error vector.

(h) T **F:** If Q is square orthogonal matrix such that $Q^{2019} = I$ then $Q^{2018} = Q^T$.

Solution:
By condition, $Q^TQ = QQ^T = I$, i.e., $Q^{-1} = Q^T$, therefore $Q^{2019} = I \Rightarrow Q^{2018}Q = I$, hence by uniqueness of the inverse $Q^{2018} = Q^{-1} = Q^T$.

(i) T **F:** If three vectors in $\mathbb{R}^3$ are orthonormal then they form a basis in $\mathbb{R}^3$.

Solution:
Since $\dim \mathbb{R}^3 = 3$, any linearly independent set of three vectors will be a basis for $\mathbb{R}^3$, and if three vectors are orthonormal, then they will form linearly independent set, therefore a basis for $\mathbb{R}^3$.

(j) **T** **F:** If columns of matrix A are orthogonal then rows of A are independent.

Solution:
Consider the counterexample $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, where is readily seen the columns are orthogonal, but the rows are not linearly independent.