MATH 2418: Linear Algebra

Assignment 8 (sections 3.4 and 3.5)

Due: March 27, 2019 Term: Spring, 2019

[First Name] [Last Name] [Net ID]

Suggested problems (do not turn in):Section 3.4: 1, 2, 3, 4, 5, 11, 12, 13, 15, 16, 17, 18, 24, 26, 27, 35; Section 3.5: 1, 2, 4, 5, 6, 9, 12, 13, 14, 18, 19, 22, 23, 26, 27; Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points]

(a) (2 points) Determine if the set of vectors $\{(1,2,3),(5,6,7),(8,9,10),(0,1,1)\}$ form a basis in $\mathbb{R}^3$.

Solution:
No. Any basis in $\mathbb{R}^3$ consists of exactly 3 vectors.

(b) (3 points) Find a subset of the above set of vectors, which form a basis in $\mathbb{R}^3$.

Solution:
Let A be a matrix of the above vectors written in columns, *i.e.*

$$A = \begin{bmatrix} 1 & 5 & 8 & 0 \\ 2 & 6 & 9 & 1 \\ 3 & 7 & 10 & 1 \end{bmatrix}.$$

Matrix A can be reduced to its row echelon form

$$A_e = \begin{bmatrix} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which has 3 pivots. Therefore, the corresponding columns of A are linearly independent, which means that vectors $\{(1,2,3),(5,6,7),(0,1,1)\}$ form a basis in $\mathbb{R}^3$.

(c) (5 points) Find a basis in the vector space set of all polynomials $p(x)$ with degree ≤ 4, satisfying the property $p(-2) = 0$. What is the dimension of this space?

Solution:
Consider an arbitrary polynomial of degree ≤ 4 $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. Then $p(-2) = a_0 - 2a_1 + 4a_2 - 8a_3 + 16a^4 = 0$. Thus, we need to find a basis in the null-space of

the matrix $M = [1, -2, 4, -8, 16]$. It is easy to see that

$$N(B) = \text{span}\{(-16, 0, 0, 0, 1), (8, 0, 0, 1, 0), (-4, 0, 1, 0, 0), (2, 1, 0, 0, 0)\}$$

. Thus, the answer is $\{x^4 - 16,\ x^3 + 8,\ x^2 - 4,\ x + 2\}$.

2. [10 points] Let V be a subspace of $\mathbb{R}^3$, spanned by the vectors $(2, 3, 1)$ and $(0, 4, 4)$.

(a) (4 pts) Construct a matrix A, such that $R(A) = C(A^T) = V$.

Solution:

For example, take matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$ whose row space R(A) is obviously spanned by (2,3,1) and (0,4,4).

Then $A^T = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 4 \\ 1 & 4 & 4 \end{bmatrix}$ with $C(A^T)$ spanned by by (2,3,1) and (0,4,4).

(b) (4 pts) Construct a matrix B such that $N(B) = V$.

Solution:

Take matrix $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ with the reduced form $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Clearly, $N(B) = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$.

Note that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \frac{3}{8} \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$,

Hence, null space of B spanned also by (2,3,1) and (0,4,4) and $N(B) = V$

(c) (2 pts) Find AB.

Solution:

$$AB = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 6 \\ 6 & -8 & 8 \\ 8 & -8 & 8 \end{bmatrix}$$

Note that matrices A and B are not unique and may be chosen differently.

3. [10 points] Find the dimensions and bases for the four fundamental subspaces of matrix

$$A = \begin{bmatrix} 2 & -3 & 4 & 5 & -6 \\ 4 & -5 & 8 & 9 & -14 \\ 6 & -10 & 12 & 16 & -16 \end{bmatrix}$$

Solution:

By performing row reducing operations, we have the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & -6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the dimensions of row space $C(A^T)$ is 2, one basis is $\{(1, 0, 2, 1, -6), (0, 1, 0, -1, -2)\}$.
The dimension of $C(A)$ is 2, one basis is the first two columns of A $\{(2, 4, 6), (-3, -5, -10)\}$.
There are three free variables x_3, x_4, x_5, and the null space is

$$N(A) = \begin{bmatrix} -2x_3 - x_4 + 6x_5 \\ x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ x_3, x_4, x_5 \in \mathbb{R}$$

The dimension of null space $N(A)$ is 3, one basis is $\{(-2, 0, 1, 0, 0), (-1, 1, 0, 1, 0), (6, 2, 0, 0, 1)\}$.
The reduced row echelon form of A^T:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N(A^T) = \begin{bmatrix} -5x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}, \ x_3 \in \mathbb{R}$$

Te dimension of left null space $N(A^T)$ is 1, one basis is $\{(-5, 1, 1)\}$.

4. [10 points] Let $\mathbf{u} = [2,3,1]$, $\mathbf{v} = [0,1,-1]$, $\mathbf{a} = [2,1]$ and $\mathbf{b} = [-3,-2]$.

   (a) (3 pts) Find the rank of the matrix $B = \mathbf{a}^T\mathbf{u} + \mathbf{b}^T\mathbf{v}$.

   (b) (3 pts) Find the dimension of $N(B)$.

   (c) (4 pts) Find a basis for $C(B)$.

Solution:

(a)

$$\begin{aligned} B &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & 2 \\ 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 3 \\ 0 & -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 5 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[R2 \mapsto R2 - \frac{1}{2}R1]{} \begin{bmatrix} 4 & 6 & 2 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Thus rank(B) = 2

(b) There is only one free variable and hence dim(N(B)) = 1

(c) The pivot columns are columns 1 and column 2 and hence Basis for $C(B)$ is $\{(4,2),(3,1)\}$

5. [10 points] Find a bases for the $C(A)$ and $R(A) = C(A^T)$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 9 & 1 \end{bmatrix}$.

Solution:

Setting $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 9 & 1 \end{bmatrix}$, we have $A = EB = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 12 & 15 & 18 \\ 6 & 21 & 35 & 32 \end{bmatrix}$. Since E is invertible, the linearly independent colums of A are in the same position as the pivot columns of B, in that case are columns 1, 2 and 3, hence a basis for $C(A)$ is $\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ 21 \end{bmatrix}, \begin{bmatrix} 4 \\ 15 \\ 35 \end{bmatrix} \right\}$. On the other side, rank(A) = rank(EB) = rank$(B) = 3$, hence all rows of A are linearly independent, therefore a basis for $C(A^T)$ is $\left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \\ 15 \\ 18 \end{bmatrix}, \begin{bmatrix} 6 \\ 21 \\ 35 \\ 32 \end{bmatrix} \right\}$

6. [10 points] True or False? Circle your answer and **provide a justification** for your choice.

   (a) **T F:** If the row space of matrix A equals to its column space then $A^T = A$.

   (b) **T F:** If A is an invertible $(m \times m)$ matrix and B is $(n \times m)$ matrix, then A and BA have the same null-space.

   (c) **T F:** Dimensions of the column-spaces for the matrices A and $\begin{bmatrix} A & A & A \\ A & A & A \end{bmatrix}$ coincide.

   (d) **T F:** Let $B = AA^T$. Then column space and row space of B coincide.

   (e) **T F:** If null-space and left null-space of matrix A coincide, then $A = A^T$.

Solution:

a) False – all invertible matrices A have $R(A) = C(A)$, yet not all of them are symmetric.

b) False – let $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $BA \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in N(BA)$. However, $B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin N(B)$, hence $N(BA) \neq N(B)$

c) True – The RREF, U, of $\begin{bmatrix} A & A & A \\ A & A & A \end{bmatrix}$ is $\begin{bmatrix} U & U & U \\ 0 & 0 & 0 \end{bmatrix}$, which has the same number of pivots as U has. Hence their dimensions coincide.

d) True – AA^T is a symmetric matrix. Hence, the set of its row vectors equals the set of its column vectors. Hence, the span of these sets must be equal, the spans being $R(AA^T)$ and $C(AA^T)$. Thus, $R(AA^T) = C(AA^T)$.

Another way of looking at this is to observe that for any matrix B, $C(B) = R(B^T)$, so $C(AA^T) = R((AA^T)^T) = R((A^T)^T A^T) = R(AA^T)$, giving us our result again. **Rule: For symmetric matrices S, $R(S) = C(S)$.**

e) False – By the same reasoning as in a), we may look at any non-symmetric invertible matrix A. In this case, $N(A) = N(A^T) = \{0\}$ yet $A^T \neq A$ (since A is **not** symmetric).