MATH 2418: Linear Algebra

Assignment 7 (sections 3.3 and 3.4)

Due: `March 13, 2019` Term: `Spring, 2019`

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Suggested problems (do not turn in): Section 3.3: 1, 2, 3, 4, 5, 7, 16, 17, 25, 26, 27, 28, 29; Section 3.4: 1, 2, 3, 4, 5, 11, 12, 13, 15, 16, 17, 18, 24, 26, 27, 35. Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points] Given linear system $\begin{cases} \quad\quad\;\; x_2 + x_3 + 3x_4 + x_5 = 0 \\ 2x_1 + 3x_2 + x_3 + \;\; x_4 \quad\quad\; = -1 \\ 6x_1 + 2x_2 \quad\quad + 6x_4 + x_5 = 1 \end{cases}$ corresponding to $A\mathbf{x} = \mathbf{b}$.

   (a) Solve the system.
   Solution: Writing the corresponding augmented matrix and applying elementary operations:

$$[A \mid \mathbf{b}] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 6 & 2 & 0 & 6 & 1 & 1 \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & -7 & -3 & 3 & 1 & 4 \end{array}\right] \xrightarrow{R_3 + 7R_2} \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 4 & 24 & 8 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 6 & 2 & 1 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccccc|c} 2 & 3 & 0 & -5 & -2 & -2 \\ 0 & 1 & 0 & -3 & -1 & -1 \\ 0 & 0 & 1 & 6 & 2 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccccc|c} 2 & 0 & 0 & 4 & 1 & 1 \\ 0 & 1 & 0 & -3 & -1 & -1 \\ 0 & 0 & 1 & 6 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 1/2 & 1/2 \\ 0 & 1 & 0 & -3 & -1 & -1 \\ 0 & 0 & 1 & 6 & 2 & 0 \end{array}\right] = [R \mid \mathbf{d}] \Rightarrow \mathbf{x}_p = \begin{bmatrix} 1/2 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

   (b) Write your solution as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is the particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_n$ is a linear combination of special solutions of $A\mathbf{x} = \mathbf{0}$.
   Solution: From $R = \textbf{rref}(A)$, we see the number of special solutions are $5 - 3 = 2$, hence we build the linear combination $\mathbf{x}_n = s\mathbf{S}_1 + t\mathbf{S}_2$, where $s, t \in \mathbb{R}$ and special solutions $\mathbf{S}_1$, $\mathbf{S}_2$ are :

$$\mathbf{S}_1 = \begin{bmatrix} -2 \\ 3 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \;\; \mathbf{S}_2 = \begin{bmatrix} -1/2 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

And finally the complete solution is $\mathbf{x} = \begin{bmatrix} 1/2 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where $s, t \in \mathbb{R}$

(c) What is the rank of the coefficient matrix A?
Solution:
From $R = \textsf{\textbf{rref}}(A)$ we observe there are 3 pivots, therefore the rank of A is 3

2. [10 points] Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Which of the spaces $C(A)$, $C(U)$, $C(A^T)$, $C(U^T)$ are the same?

$$C(A) = \operatorname{span}\{(1,1,4),(1,2,5),(0,3,3)\} = \operatorname{span}\{(1,1,4),(0,1,1)\} = \operatorname{span}\{(1,0,3),(0,1,1)\}$$

$$C(U) = \operatorname{span}\{(1,0,0),(1,1,0),(0,3,0)\} = \operatorname{span}\{(1,0,0),(1,1,0)\} = \operatorname{span}\{(1,0,0),(0,1,0)\}$$

$$C(A^T) = \operatorname{span}\{(1,1,0),(1,2,3),(4,5,3)\} = \operatorname{span}\{(1,1,0),(1,2,3)\} = \operatorname{span}\{(1,1,0),(0,1,3)\}$$

$$C(U^T) = \operatorname{span}\{(1,1,0),(0,1,3),(0,0,0)\} = \operatorname{span}\{(1,1,0),(0,1,3)\}$$

Thus we have that $C(A^T) = C(U^T)$.

3. [10 points] Find the LU factorization of A and the complete solution to $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution:

Reduce the augmented system into RREF as:

$$[A \mid \mathbf{b}] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus LU factorization is: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

x_3, x_4 are free variables. The particular solution is $\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

The complete solution is: $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

4. [10 points] Determine if the given vectors form a basis for the vector space specified.

(a) $V = M_{22}$ (the vector space of all 2×2 matrices), given set of vectors

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

Solution:
The linear combination of these vectors is

$$c_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -c_1 + 2c_2 + c_3 & 0 \\ c_2 + 2c_4 & c_3 + c4 \end{bmatrix}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.
Notice that the upper right corner component is always zero, which means the given set of vectors cannot span the vector space M_{22}, therefore the given set is not a basis.

(b) $V = M_{22}$ (the vector space of all 2×2 matrices), given set of vectors

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:
The linear combination of these vectors is

$$c_1 \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 + c_3 + c_4 & -c_2 - c_3 \\ c_1 + 2c_2 + c_3 & c_1 + c_3 + c_4 \end{bmatrix}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.
To check the linear independence of these vectors, we let the combination equals to zero and we have

$$\begin{cases} 2c_1 + c_2 + c_3 + c_4 = 0 \\ -c_2 - c_3 = 0 \\ c_1 + 2c_2 + c_3 = 0 \\ c_1 + c_3 + c_4 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \\ c_3 = 1 \\ c_4 = -2 \end{cases}$$

The nonzero solution implies that the given vectors are **not** linearly independent, so the given set is not a basis.

(c) $V = P_2$(the vector space of all polynomials of degree ≤ 2), given set of vectors

$$4 + x,\ x^2 - x - 1,\ x^2 + x - 3.$$

Solution:
The linear combination of these vectors is

$$c_1(4 + x) + c_2(x^2 - x - 1) + c_3(x^2 + x - 3) = (c_2 + c_3)x^2 + (c_1 - c_2 + c_3)x + (4c_1 - c_2 - 3c_3)$$

where $c_1, c_2, c_3 \in \mathbb{R}$.
To check the linear independence of these vectors, we let the combination equals to zero and we have

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 - c_2 + c_3 = 0 \\ 4c_1 - c_2 - 3c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

Hence, these vectors are linearly independent. Since c_1, c_2, c_3 can be any real numbers, these vectors also span the vector space P_2. Therefore, the given set is a basis.

5. [10 points] Let $\mathbf{v}_1 = (-2,-7,2)$, $\mathbf{v}_2 = (5,1,-5)$, $\mathbf{v}_3 = (-4,-9,4)$, $\mathbf{v}_4 = (-3,-6,3)$, $\mathbf{v}_5 = (1,2,-1)$. Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ that forms a basis for the subspace of $\mathbb{R}^3$ spanned by those five vectors. Express each non-basis vector as a linear combination of basis vectors.

Solution. Consider a matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5]$. We need to find a basis in the column space $C(A)$. In the reduced row echelon form A becomes

$$A_{rref} = \begin{bmatrix} 1 & 0 & \frac{41}{33} & \frac{9}{11} & -\frac{3}{11} \\ 0 & 1 & -\frac{10}{33} & -\frac{3}{11} & \frac{1}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Two pivots are in the first and the second columns. Thus, vectors $\mathbf{v}_1, \mathbf{v}_2$ form a basis in $C(A)$. Clearly,

$$\mathbf{v}_3 = \frac{41}{33}\mathbf{v}_1 - \frac{10}{33}\mathbf{v}_2, \tag{1}$$

$$\mathbf{v}_4 = \frac{9}{11}\mathbf{v}_1 + \frac{3}{11}\mathbf{v}_2, \tag{2}$$

$$\mathbf{v}_5 = -\frac{3}{11}\mathbf{v}_1 + \frac{1}{11}\mathbf{v}_2. \tag{3}$$

6. [10 points] Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be three linearly independent vectors in $\mathbb{R}^3$.

(a) Find the rank of the matrix $A = \begin{bmatrix}(\mathbf{v}_1 - \mathbf{v}_2) & (\mathbf{v}_2 - \mathbf{v}_3) & (\mathbf{v}_3 - \mathbf{v}_1)\end{bmatrix}$.

(b) Find the rank of the matrix $B = \begin{bmatrix}(\mathbf{v}_1 + \mathbf{v}_2) & (\mathbf{v}_2 + \mathbf{v}_3) & (\mathbf{v}_3 + \mathbf{v}_1)\end{bmatrix}$.

Solution(a):

Since $(v_1 - v_2) + (v_2 - v_3) + (v_3 - v_1) = v_1 + v_2 + v_3 - v_1 - v_2 - v_3 = 0$
So the vectors $(v_1 - v_2), (v_2 - v_3)$ and $(v_3 - v_1)$ are linearly dependent.
Also $(v_1 - v_2) + (v_2 - v_3) = (v_3 - v_1)$ implies that the vector $(v_3 - v_1)$ lies on the same plane as $(v_1 - v_2), (v_2 - v_3)$
Moreover, the vectors $(v_1 - v_2), (v_2 - v_3)$ are linearly independent. Therefore, the matrix A has two independent columns. So the rank of $A = 2$.

Solution(b):

Here, $c_1(v_1 + v_2) + c_2(v_2 + v_3) + c_3(v_3 + v_1) = 0$ where $c_1, c_2, c_3 \in \mathbb{R}$
$\implies (c_1 + c_3)v_1 + (c_1 + c_2)v_2 + (c_2 + c_3)v_3 = 0$
$\implies (c_1 + c_3) = (c_1 + c_2) = (c_2 + c_3) = 0$, since v_1, v_2, v_3 are linearly independent vectors.
$\implies (c_1 + c_3) = 0$ or, $c_1 = -c_3$ and $(c_1 + c_2) = 0$ or, $c_1 = -c_2 \implies c_2 = c_3$
Also $(c_2 + c_3) = 0 \implies c_2 = c_3 = 0 \implies c_1 = 0$
Which implies the linear independence of the vectors $(v_1 + v_2),\ (v_2 + v_3),\ (v_3 + v_1)$
Hence, rank of the matrix $B = 3$

7. [10 points] Prove that $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right\}$ is a basis for $U_{2\times 2}$, the vector space of all 2×2 upper triangular real matrices.

Solution: <u>Linear independence</u>

$$
\begin{aligned}
x \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} x+y+z & x+2y+z \\ 0 & x+2z \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow x + 2z = 0 &\Leftrightarrow x = -2z \\
\Rightarrow x + y + z = y - z = 0 \text{ and } x + 2y + z &= 2y - z = 0 \\
\Rightarrow y = z = 0 &\Rightarrow x = 0
\end{aligned}
$$

Thus the set B consist of a linearly independent matrices.

<u>Span</u>

Let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ be any upper triangular real matrix. Then

$$
\begin{aligned}
x \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \\
\begin{bmatrix} x+y+z & x+2y+z \\ 0 & x+2z \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \\
\Rightarrow x + 2z = c &\Leftrightarrow x = c - 2z \\
\Rightarrow x + y + z = c + y - z = a \text{ and } x + 2y + z = c + 2y - z &= y + a = b \\
\Rightarrow y = b - a \Rightarrow z = c + b - 2a \text{ and } x &= 4a - 2b - c
\end{aligned}
$$

Thus for any upper triangular 2×2 real matrix there exists scalars such that the upper triangular matrix is a linear combinations of the matrices in B. Thus the set B spans the set of all 2×2 upper triangular real matrices

Therefore, the set B is a basis.

8. [10 points] Find a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v_1} = (0,2,6)$, $\mathbf{v_2} = (1,3,2)$, $\mathbf{v_3} = (4,1,4)$, $\mathbf{v_4} = (3,1,6)$, $\mathbf{v_5} = (1,0,1)$. Express each non-basis vector as a linear combination of basis vectors.

Solution:

Let $A = \begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} & \mathbf{v_5}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 3 & 1 \\ 2 & 3 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 & 1 \end{bmatrix}$. The pivot columns of A are a basis for its column space.

The $\mathbf{ref}(A)$ it's given by:

$$\begin{bmatrix} 0 & 1 & 4 & 3 & 1 \\ 2 & 3 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 4 & 3 & 1 \\ 6 & 2 & 4 & 6 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 4 & 3 & 1 \\ 0 & -7 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 + 7R_2} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 4 & 3 & 1 \\ 0 & 0 & 29 & 24 & 8 \end{bmatrix}$$

The first 3 columns are the pivot columns. Then the corresponding three columns from A: $\mathbf{v_1} = (0,2,6)$, $\mathbf{v_2} = (1,3,2)$, and $\mathbf{v_3} = (4,1,4)$ are the elements of the basis for the subspace of $\mathbb{R}^3$.

Now, $\mathbf{v_4}$ and $\mathbf{v_5}$ are the non-basis vectors, to express them as a linear combination of basis vector, we do back substitution form the $\mathbf{ref}(A)$:

For $\mathbf{v_4}$, we have $29z = 24$, so $z = \frac{24}{29}$, $y + 4x = 3$, so $y = 3 - 4z = -\frac{9}{29}$, and $2x + 3y + x = 1$, so $x = \frac{16}{29}$. Then

$$\mathbf{v_4} = \mathbf{x}\ \mathbf{v_1} + \mathbf{y}\mathbf{v_2} + \mathbf{z}\mathbf{v_3} = \mathbf{\frac{16}{29}}\mathbf{v_1} - \mathbf{\frac{9}{29}}\mathbf{v_2} + \mathbf{\frac{24}{29}}\mathbf{v_3}$$

For $\mathbf{v_5}$:

$29z = 8$, so $z = \frac{8}{29}$, $y + 4x = 1$, so $y = 1 - 4z = -\frac{3}{29}$, and $2x + 3y + x = 0$, so $x = \frac{1}{58}$. Then

$$\mathbf{v_5} = \mathbf{x}\mathbf{v_1} + \mathbf{y}\mathbf{v_2} + \mathbf{z}\mathbf{v_3} = \mathbf{\frac{1}{58}}\mathbf{v_1} - \mathbf{\frac{3}{29}}\mathbf{v_2} + \mathbf{\frac{8}{29}}\mathbf{v_3}$$

9. [10 points] (a) Determine if the vectors $2-x^2,\ 1+x,\ 1+2x$ form a basis for P_2(the vector space of all polynomials of degree ≤ 2).

Solution: Observe that $(1+2x)-(1+x)=x$, $2(1+x)-(1+2x)=1$, and $-(2-x^2)+4(1+x)-2(1+2x)=x^2$, so $1,x,x^2 \in span\{1+x,1+2x,2-x^2\}$. Since $\{1,x,x^2\}$ is a basis of P_2 and $\{1+x,1+2x,2-x^2\}$ is a set of three polynomials whose span contains the basis of P_2, we may conclude that $\{1+x,1+2x,2-x^2\}$ forms a basis for P_2.

(b) Determine if the vectors $(1,3,2),\ (0,2,6),\ (4,1,4)$ form a basis for $\mathbb{R}^3$.

Solution: Let $A=\begin{bmatrix}1&3&2\\0&2&6\\4&1&4\end{bmatrix}$. If we row reduce A to an upper triangular matrix with three pivots, then the rows (and columns) of A will be linearly independent and will hence form a basis of $\mathbb{R}^3$. Else, they will be dependent and so will not form a basis.

$A=\begin{bmatrix}1&3&2\\0&2&6\\4&1&4\end{bmatrix} \xrightarrow{r_3-4r_1} \begin{bmatrix}1&3&2\\0&2&6\\0&-11&-4\end{bmatrix} \xrightarrow{r_3+11/2r_2} \begin{bmatrix}1&3&2\\0&2&6\\0&0&29\end{bmatrix}$, which has three pivots, showing that our three vectors form a basis of $\mathbb{R}^3$.

10. [10 points] Consider the plane P represented by the equation $2x + 3y - 2z = 0$.

   (a) Find a basis for P.

   (b) Find a basis for the intersection of P with $yz-$plane.

Solution

(a) Note that P is the null-space of matrix $A = [2 \;\; 3 \;\; -2]$. Treating y and z as free variables and solving $2x + 3y - 2z = 0$ for x yields,

$$v \in N(A) \quad \Longleftrightarrow \quad v = \begin{bmatrix} -\frac{3}{2}y + z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad y, z \in \mathbb{R}.$$

Therefore, a basis for P is $\left\{ [-3/2 \;\; 1 \;\; 0]^T, \; [1 \;\; 0 \;\; 1]^T \right\}$.

(b) Intersection of P with yz-plane is defined by the following system of equations

$$\begin{aligned} x &= 0, \\ 2x + 3y - 2z &= 0. \end{aligned}$$

In other words. we need to find null-space of matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \end{bmatrix}.$$

Note that,

$$u \quad \in N(B) \quad \Longleftrightarrow u = t \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus, a basis for the intersection of P with yz-plane is $\left\{ [0 \;\; 2 \;\; 3]^T \right\}$.