MATH 2418: Linear Algebra

Assignment 6 (sections 3.1 and 3.2)

Due: `March 06, 2019` Term: `Spring, 2019`

[First Name] [Last Name] [Net ID]

Suggested problems (do not turn in): Section 3.1: 1, 2, 5, 9, 10, 11, 12, 19, 20, 24, 26; Section 3.2: 1,2, 3, 4, 8, 12, 15, 18, 31. Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points] Find the nullspace of $A = \begin{bmatrix} 0 & 1 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 6 & 1 \end{bmatrix}$. What is rank of A? Also find the special solutions of $A\mathbf{x} = \mathbf{0}$.

Solution:

Firstly, let's calculate the RREF for A by applying elementary row operations:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 6 & 2 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & -7 & -3 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 + 7R_2} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 4 & 24 & 8 \end{bmatrix} \xrightarrow{\frac{1}{4}R_3}$$

$$\xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 6 & 2 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_3} \begin{bmatrix} 2 & 3 & 0 & -5 & -2 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 6 & 2 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 6 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 & 2 & 1/2 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 6 & 2 \end{bmatrix}$$

From $R = \textbf{rref}(A)$, it is immediately seen that R has 3 pivots, hence its rank is 3, therefore rank of A is 3 also. The nullspace of A, denoted by $N(A)$, is the set of all linear combinarions of the special solutions to $A\mathbf{x} = \mathbf{0}$. Since A has 5 columns and 3 pivots, the number of special solutions is $5 - 3 = 2$ and they are:

$$\mathbf{S_1} = \begin{bmatrix} -2 \\ 3 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{S_2} = \begin{bmatrix} -1/2 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

2. [10 points] (a) Suppose matrix A reduces into echelon form U, prove that $N(A) = N(U)$.

Solution:

Let $U = MA$, then let $y \in N(A)$. We have that:

$$Uy = (MA)y = M(Ay) = M(0) = 0$$

So we have that $y \in N(U)$ and thus $N(A) \subseteq N(U)$.

Now take a $x \in N(U)$, this means that:

$$Ux = 0 \Rightarrow M^{-1} * (Ux) = M^{-1}(0) = 0 \Rightarrow M^{-1}(MAx) = 0 \Rightarrow Ax = 0$$

So we have that $x \in N(A)$ and thus $N(U) \subseteq N(A)$. Therefore $N(A) = N(U)$.

(b) Write a 2×2 matrix A such that $C(A) \neq C(U)$, where U is the echelon form of matrix A.

Solution:

Define:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Then we have that the reduced echelon form of matrix A is:

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Note that:

$$C(A) = \text{span}\{(1, 2)\}$$

$$C(U) = \text{span}\{(1, 0)\}$$

Thus $C(A) \neq C(U)$.

3. [10 points] (a) Determine if the vectors $\mathbf{v}_1 = (1, 2, -1)$, $\mathbf{v}_2 = (3, 8, 0)$, $\mathbf{v}_3 = (1, 1, 1)$ span $\mathbb{R}^3$.

Solution:

Yes. Consider the matrix determined by the three vectors. The upper triangular matrix U is:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 - 3R_1]{R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 + 0.5R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 3.5 \end{bmatrix} = U$$

Since there are three nonzero pivots, so they are linearly independent in $\mathbb{R}$, thus they span $\mathbb{R}$.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$. If yes, write $\mathbf{b}$ as a linear combination of columns of A.

Solution:

Yes. We compute the solution for $A\mathbf{x} = \mathbf{b}$ by reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 2 & 8 & 1 & 4 \\ -1 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 3 & 2 & 5 \end{array}\right] \xrightarrow{R_3 - 1.5R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 3.5 & 8 \end{array}\right]$$

By back substitution, we have $x_3 = \frac{16}{7}, x_2 = \frac{1}{7}, x_1 = \frac{2}{7}$. Thus $\mathbf{b}$ is in the column space of A and the linear combination is:

$$\mathbf{b} = \frac{2}{7} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + \frac{16}{7} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. [10 points] Determine if the set consisting of

(a) (2 pts) all $(x,y,z) \in \mathbb{R}^3$ with $x=-z$ is a subspace of $\mathbb{R}^3$
Solution:
Let $W = \{(x,y,z) \in \mathbb{R}^3 | x=-z\}$. Then for any pair $(x_1,y_1,z_1),(x_2,y_2,z_2) \in W$ (note that $x_1=-z_1$, $x_2=-z_2$) and $c,d \in \mathbb{R}$ we have

$$\begin{aligned} c(x_1,y_1,z_1)+d(x_2,y_2,z_2) &= (cx_1+dx_2, cy_1+dy_2, cz_1+dz_2) \\ &= (cx_1+dx_2, cy_1+dy_2, -cx_1-dx_2) \\ &= (cx_1+dx_2, cy_1+dy_2, -(cx_1+dx_2)) \end{aligned}$$

Thus we have that W is a subspace of $\mathbb{R}^3$ since $c(x_1,y_1,z_1)+d(x_2,y_2,z_2) \in W$ for any pair of points in W and any $c,d \in \mathbb{R}$.

(b) (2 pts) all $(x,y,z) \in \mathbb{R}^3$ with $x=-z-2$ is a subspace of $\mathbb{R}^3$
Solution:
Let $W = \{(x,y,z) \in \mathbb{R}^3 | x=-z-2\}$. Then for any pair $(x_1,y_1,z_1),(x_2,y_2,z_2) \in W$ (note that $x_1=-z_1-2$, $x_2=-z_2-2$) and $c,d \in \mathbb{R}$ we have

$$\begin{aligned} c(x_1,y_1,z_1)+d(x_2,y_2,z_2) &= (cx_1+dx_2, cy_1+dy_2, cz_1+dz_2) \\ &= (cx_1+dx_2, cy_1+dy_2, -c(x_1-2)-d(x_2-2)) \\ &= (cx_1+dx_2, cy_1+dy_2, -(cx_1+dx_2)-2(c+d)) \end{aligned}$$

Since $2(c+d) \neq 2$ for all $c,d \in \mathbb{R}$ we have that $c(x_1,y_1,z_1)+d(x_2,y_2,z_2) \notin W$ and hence W is not a subspace of $\mathbb{R}^3$.

(c) (3 pts) all vectors $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x}=\mathbf{0}$ where A is an $n \times n$ real matrix, is a subspace of $\mathbb{R}^n$.
Solution:
Let $W = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x}=\mathbf{0}\}$. Then for any pair $\mathbf{x}_1,\mathbf{x}_2 \in W$ and $c,d \in \mathbb{R}$ we have

$$A(c\mathbf{x}_1+d\mathbf{x}_2) = cA\mathbf{x}_1+dA\mathbf{x}_2 = c\cdot\mathbf{0}+d\cdot\mathbf{0}=\mathbf{0}$$

Thus we have that W is a subspace of $\mathbb{R}^n$ since $c\mathbf{x}_1+d\mathbf{x}_2 \in W$ for any pair of points in W and any $c,d \in \mathbb{R}$.

(d) (3 pts) $D_{2\times2} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times2}$, the vector space of all 2×2 real matrices.
Solution:
For any pair $A_1,A_2 \in D_{2\times2}$ where $A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ and $c,d \in \mathbb{R}$ we have

$$cA_1+dA_2 = c\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + d\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} ca_1+da_2 & 0 \\ 0 & cb_1+db_2 \end{bmatrix} \in D_{2\times2}$$

Thus we have that $D_{2\times2}$ is a subspace of $M_{2\times2}$ since $cA_1+dA_2 \in D_{2\times2}$ for any pair of points in $D_{2\times2}$ and any $c,d \in \mathbb{R}$.

5. [10 points] Determine if column space of the matrix $A = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 9 & 9 & 8 \\ 1 & 1 & 1 & 6 & 7 & 1 \\ 7 & 8 & 9 & 9 & 0 & 2 \end{bmatrix}$ contains the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 7 \\ 6 \end{bmatrix}$.

Recall that b is contained in column space of A if it can be represented as a linear combination of columns A:

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 9 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 9 \\ 6 \\ 9 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 9 \\ 7 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 7 \\ 6 \end{bmatrix}$$

The respective row reduced form of the augmented matrix corresponding to this equation is:

$$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 9 & 9 & 8 & 7 \\ 1 & 1 & 1 & 6 & 7 & 1 & 7 \\ 7 & 8 & 9 & 9 & 0 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & \frac{827}{18} & -\frac{163}{9} \\ 0 & 1 & 2 & 0 & 0 & -\frac{413}{9} & \frac{164}{9} \\ 0 & 0 & 0 & 1 & 0 & \frac{95}{18} & -\frac{13}{9} \\ 0 & 0 & 0 & 0 & 1 & -\frac{79}{18} & \frac{20}{9} \end{bmatrix}$$

Hence, we can find scalars $x_1, x_2, x_3, x_4, x_5, x_6$ which represent b as a linear combination of columns A. Therefore b is in column space of A.

[10] Find reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 4 & -5 \\ 2 & 3 & -6 & 7 & 9 \\ -2 & 3 & 6 & -8 & 10 \\ 1 & 2 & -3 & 4 & 6 \end{bmatrix}$. Which variables are free?
Solution:

The given matrix is: $A = \begin{bmatrix} 1 & 2 & -3 & 4 & -5 \\ 2 & 3 & -6 & 7 & 9 \\ -2 & 3 & 6 & -8 & 10 \\ 1 & 2 & -3 & 4 & 6 \end{bmatrix}$

Let us apply the following row operations: $R_2' = R_2 - 2R_1, R_3' = R_3 + 2R_1, R_4' = R_4 - R_1$:

Then A reduces to $\begin{bmatrix} 1 & 2 & -3 & 4 & -5 \\ 0 & -1 & 0 & -1 & 19 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix} \xrightarrow[R_1'=R_1+2R_2]{R_3'=R_3+7R_2} \begin{bmatrix} 1 & 0 & -3 & 2 & 33 \\ 0 & -1 & 0 & -1 & 19 \\ 0 & 0 & 0 & -7 & 133 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix}$

$\xrightarrow[R_2'=-R_2]{R_3'=\frac{-1}{7}R_3} \begin{bmatrix} 1 & 0 & -3 & 2 & 33 \\ 0 & 1 & 0 & 1 & -19 \\ 0 & 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix} \xrightarrow[R_2'=R_2-R_3]{R_1'=R_1-2R_3} \begin{bmatrix} 1 & 0 & -3 & 0 & 71 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix}$

$\xrightarrow[R_4'=\frac{1}{11}R_4]{} \begin{bmatrix} 1 & 0 & -3 & 0 & 71 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1'=R_1-71R_4]{R_3'=R_3+19R_4} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Therefore, the row reduced form of A is :

rref(A)= $\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The variable corresponding to column 3 i.e x_3 is free.

[10] Given $A = \begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 2 & -4 & 6 & -1 & 1 & 3 \\ 3 & -6 & 9 & -1 & 2 & 1 \\ -4 & 8 & -12 & 2 & -2 & -3 \end{bmatrix}$

(a) Find the nullspace $N(A)$.

Solution:

$$\begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 2 & -4 & 6 & -1 & 1 & 3 \\ 3 & -6 & 9 & -1 & 2 & 1 \\ -4 & 8 & -12 & 2 & -2 & -3 \end{bmatrix} \xrightarrow[R_2 \mapsto R_2 - 2R_3]{R_3 \mapsto R_3 - 3R_1} R_4 \mapsto R_4 + 4R_1 \begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 5 & 5 & -5 \\ 0 & 0 & 0 & 6 & -6 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 \mapsto R_3 - \frac{5}{3}R_2]{R_4 \mapsto R_4 + 2R_2} \begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} R_4 \mapsto R_4 + \frac{9}{11}R_3 \begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the free variables are x_2, x_3, x_5 and hence be solve the system $A\boldsymbol{x} = 0$ by setting $x_2 = 1, x_3 = 0 = x_5$ and $x_2 = 0 = x_5, x_3 = 1$ and $x_2 = 0 = x_3, x_5 = 1$ For

$$x_2 = 1, x_3 = x_5 = 0$$

$$\begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_4 \\ 0 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by back substitution gives

$$\begin{aligned} x_6 &= 0 \\ x_4 &= 0 \\ x_1 - 2 = 0 \Leftrightarrow x_1 &= 2 \\ \boldsymbol{s_1} &= (2, 1, 0, 0, 0, 0) \end{aligned}$$

For $x_2 = 0 = x_5, x_3 = 1$

$$\begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 1 \\ x_4 \\ 0 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow x_6 &= 0, \\ x_4 &= 0, \\ x_1 &= -3 \\ \Rightarrow \boldsymbol{s}_2 &= (-3, 0, 1, 0, 0, 0) \end{aligned}$$

For $x_2 = 0 = x_3, x_5 = 1$

$$\begin{bmatrix} 1 & -2 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \\ 1 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow x_6 = 0, \\ &x_4 = -1, \\ x_1 + 2 - 1 = 0 &\Rightarrow x_1 = -1 \\ &\Rightarrow \boldsymbol{s}_3 = (-1, 0, 0, -1, 1, 0) \end{aligned}$$

Therefore the null space is given by

$$N(A) = \{\sum_{i=1}^{3} c_i \boldsymbol{s}_i | c_i \in \mathbb{R}, \text{and } \boldsymbol{s_1} = (2, 1, 0, 0, 0, 0), \boldsymbol{s}_2 \ = (-3, 0, 1, 0, 0, 0), \boldsymbol{s}_3 = (-1, 0, 0, -1, 1, 0)\}$$

(b) Find three special solutions of $A\mathbf{x} = \mathbf{0}$.

solution: From the computations above, we have the special solutions to $A\mathbf{x} = \mathbf{0}$ given

$$\begin{aligned} \boldsymbol{s_1} &= (2, 1, 0, 0, 0, 0), \\ \boldsymbol{s}_2 &= (-3, 0, 1, 0, 0, 0), \\ \boldsymbol{s}_3 &= (-1, 0, 0, -1, 1, 0) \end{aligned}$$

(c) What is the rank of A?

solution: The rank of A is given by the number of pivots and since there are 3 pivots we have $rank(A) = 3$

[10] Is the vector $\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$? Explain your answer.

Solution:

Let $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 8 \end{bmatrix}$. If the linear system $A\mathbf{x} = \mathbf{b}$ is solvable, then the vector $\mathbf{b}$ is in the column space of A, and then $\mathbf{b}$ is a linear combination of the column vectors of A.

By reducing the augmented matrix to upper form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & 8 & 3 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 7 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & 1 \end{array}\right]$$

Since we have full set of pivots (3 pivots), so the linear system is solvable and has unique solution. Therefore the vector $\mathbf{b}$ is a linear combination of columns of A.

[10] Answer the followings(you do not need to show your work).

(a) Write 1×3 matrix A whose null space is the plane $4x - 5y + 6z = 0$

Solution: Let $A = \begin{bmatrix} 4 & -5 & 6 \end{bmatrix}$. Then $N(A)$ consists of all vectors (x, y, z) satisfying $4x - 5y + 6z = 0$, which is precisely the desired plane.

(b) Write down a matrix A such that $N(A)$ is the set of all linear combinations of $(2, 0, 1, 7)$ and $(2, 0, 1, 8)$

Solution: Let's observe that the set of all linear combinations of $\mathbf{v} = (2, 0, 1, 7)$ and $\mathbf{w} = (2, 0, 1, 8)$ is the same as all linear combinations of $(2, 0, 1, 7)$ and $(0, 0, 0, 1)$ (subtracting the first from the second), which is also the same as $(2, 0, 1, 0)$ and $(0, 0, 0, 1)$. So let $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then $A\mathbf{v} = 0$ and $A\mathbf{w} = 0$, so $\mathbf{v}$ and $\mathbf{w}$ are in $N(A)$. Furthermore, the dimension of $N(A)$ is two and $\mathbf{v}$ and $\mathbf{w}$ are not collinear, so their span is $N(A)$.

(c) Construct a matrix A whose column space contains $\mathbf{v} = (-3, 0, 3)$ and $\mathbf{w} = (1, 1, 1)$ and the nullspace contains $(1, 2, 3)$.

Solution: If such an A exists, it must be 3x3 and its column space must be linear combinations of $\mathbf{v}$ and $\mathbf{w}$. If we let the first two columns be $\mathbf{v}$ and $\mathbf{w}$, respectively, then the third column, $\mathbf{c}_3$, must satisfy $\mathbf{v} + 2\mathbf{w} + 3\mathbf{c}_3 = 0$. So let $\mathbf{c}_3 = -\mathbf{v} - 2\mathbf{w}$ and $A = \begin{bmatrix} \mathbf{v} & \mathbf{w} & \mathbf{c}_3 \end{bmatrix}$. Then $C(A)$ contains (and is spanned by) $\mathbf{v}$ and $\mathbf{w}$, and satisfies $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{0}$.

(d) Construct a 2×2 matrix whose null space equals to its column space.

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. The column space is spanned by $(1, 1)$ and the null space is also spanned by $(1, 1)$.

[10]
True or False? Circle your answer and **provide a justification** for your choice.

(a) **T F:** Intersection of two planes in $\mathbb{R}^3$ is a subspace in $\mathbb{R}^3$.

(b) **T F:** Set of all singular 2×2 matrices form a subspace in M_{22}.

(c) **T F:** An invertible matrix has no free variables.

(d) **T F:** Planes $2x + 3y - z = 2018$ and $-4x - 6y + 2z = 1$ are parallel.

(e) **T F:** Matrices $\begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -5 \\ -1 & -2 & -4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ have the same null space.

Solution.

(a) False. The statement is true only if both planes pass through the origin.

(b) False. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Obviously A, B are singular, however,

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not.

(c) True. An invertible $n \times n$ matrix A has rank n. The number of free variables is $n - rank(A) = 0$.

(d) True, because

$$\begin{bmatrix} -4 \\ -6 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

(e) True. Both matrices in the reduced row echelon form are equal to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$