Assignment #5 - Spring/2019

Assignment 5 (sections 2.5, 2.6, 2.7, 3.1)

Due: `February 27, 2019` Term: `Spring, 2019`

[First Name] [Last Name] [Net ID]

Suggested problems (do not turn in):
Section 2.5: 1, 5, 6, 7, 10, 11, 12, 13, 18, 22, 25, 27, 29, 44.
Section 2.6: 1, 3, 4, 6, 8, 9, 10, 13, 14, 17, 22, 23.
Section 2.7: 1,2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 20, 21, 22, 23.
Section 3.1: 1, 2, 5, 9, 10, 11,12, 19, 20, 24, 26.
Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points] Use the Gauss-Jordan method to find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

Solution:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+6R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

2. [10 points] Consider $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$.

(a) (2 points). Use elementary row operations to reduce A in to I.

$$\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} \xrightarrow[E_{21}]{R_2 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \xrightarrow[E_{12}]{R1 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow[S_2]{R2 \longrightarrow \frac{1}{2}R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) (2 points). List all corresponding elementary matrices.

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$E_{12} = \begin{bmatrix} 1 & \frac{-3}{2} \\ 0 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

(c) (3 points). Find A^{-1} as a product of elementary matrices.

$$S_2 E_{12} E_{21} A = I \implies A^{-1} = S_2 E_{12} E_{21}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{-3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & \frac{-3}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

(d) (3 points). Express A as a product of elementary matrices.

$$S_2 E_{12} E_{21} A = I \implies A = E_{21}^{-1} E_{12}^{-1} S_2^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

3. [10 points] Solve the following problems and justify your answers by showing your work.

(a) (2 points). Give example of 2×2 non-zero matrices A, B, C such that $AB = AC$ but $B \neq C$.

Solution:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, then $AB = AC = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

(b) (2 points). Write 2×2 invertible matrices A and B such that $A + B$ is not invertible.

Solution:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is not invertible.

(c) (2 points). Write 3×3 singular matrices A and B such that $A - B$ is non-singular.

Solution:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then $A - B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is not singular.

(d) (2 points). **T** or **F**? (Circle your answer) If A and B are invertible matrices of same size, then AB and BA are both invertible.

Solution:

True.
Let A^{-1}, B^{-1} be the inverse matrices of A, B, then $(B^{-1}A^{-1})AB = I$ and $(A^{-1}B^{-1})BA = I$, thus AB and BA are both invertible.

(e) (2 points). **T** or **F**? (Circle your answer) If A^2 is not invertible, then A is not invertible.
Solution:

True.
The contrapositive is true since A is invertible, $A^{-1}A^{-1}AA = I$ gives A^2 is also invertible.

4. [10 points] Find LDU decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & -4 & 7 \end{bmatrix}$$

Solution:
Elimination process:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & -4 & 7 \end{bmatrix} \xrightarrow[l_{31}=-2]{l_{21}=-1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & -6 & 9 \end{bmatrix} \xrightarrow{l_{32}=-3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = U_1$$

$$U_1 = D \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) $L = \underline{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}}$

(b) $D = \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$

(c) $U = \underline{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$

5. [10 points] Solve the system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 1 \\ 3 & -7 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

using the following steps:

(a) (3 points). Compute factorization $A = LU$.
First, find upper triangular matrix U:

$$\begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 1 \\ 3 & -7 & 2 \end{bmatrix} \xrightarrow[R_3 + 4R_2,\, R_3 + 4R_2]{R_2 + 2R_1,\, R_3 - 3R_1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Multiplying LU produces A

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 1 \\ 3 & -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

(b) (3 points). Solve $L\mathbf{y} = \mathbf{b}$ by forward substitution.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{cases} y_1 = 1 \\ -2y_1 + y2 = -1 \\ 3y_1 - 4y_2 + y_3 = 1 \end{cases} \rightarrow \begin{cases} y_1 = 1 \\ y_2 = 1 \\ y_3 = 2 \end{cases}$$

(c) (3 points). Solve $U\mathbf{x} = \mathbf{y}$ by backward substitution.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_2 - x_3 = 1 \\ x_2 - x3 = 1 \\ x_3 = 2 \end{cases} \rightarrow \begin{cases} x_1 = 6 \\ x_2 = 3 \\ x3 = 2 \end{cases}$$

(d) (1 point). What is the solution of $A\mathbf{x} = \mathbf{b}$?

$\begin{cases} x_1 = 6 \\ x_2 = 3 \\ x3 = 2 \end{cases}$ from part (c) is the solution of both $U\mathbf{x} = \mathbf{y}$ and $A\mathbf{x} = \mathbf{b}$.

6. [10 points] Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to the system $R\mathbf{x} = \mathbf{d}$. If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? (That is, find E such that $R = EA$ and $E\mathbf{b} = \mathbf{d}$).
Solution:

Here, $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

Hence, the matrix that connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$ is $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

7. [10 points] Given matrix $A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 3 \\ -3 & -7 & 8 \end{bmatrix}$,

(a) (5 points). Show that A has no LU decomposition.
Solution:
The first pivot of the matrix A is 0 and hence A = LU decomposition fails as we require a row interchange to create a nonzero pivot.

(b) (5 points). Find the decomposition $PA = LU$, where P is an elementary permutation matrix.
Solution:
P_{12} interchanges row R_1 and R_2

$$P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 3 \\ -3 & -7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ -3 & -7 & 8 \end{bmatrix}$$

E_{31} adds 3 times R_1 to R_3

$$E_{31}P_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ -3 & -7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & -1 & 17 \end{bmatrix}$$

E_{32} adds $\frac{1}{3}$ times R_2 to R_3

$$E_{32}E_{31}P_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & -1 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & \frac{47}{3} \end{bmatrix}$$

Thus we have

$$PA = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & \frac{47}{3} \end{bmatrix}$$

8. [10 points] Let $S = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 8 \\ 2 & 8 & 23 \end{bmatrix}$ be a symmetric matrix. Find the symmetric factorization of S as $S = LDL^{\mathbf{T}}$.

Solution:

Since S is symmetric, we will find the multipliers and the pivots only. The upper triangular matrix U is:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 8 \\ 2 & 8 & 23 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & 6 & 19 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix} = U$$

So, $l_{21} = 1, l_{31} = 2$ and $l_{32} = 3$. Also pivot1=1, pivot2=2 and pivot3=1. Then:

$$S = LDL^{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

9. [10 points] (A) True or False. Circle your answer and justify it by showing your work (4 points).

(a) **T** **F**: Let A be any square matrix, then $A^{\mathbf{T}}A$, $AA^{\mathbf{T}}$, and $A+A^{\mathbf{T}}$ are all symmetric.
Solution: True. To check for symmetry, take the transpose:
$(A^TA)^T = A^T(A^T)^T = A^TA$, where we used the fact that a transpose of a product is the *reversed* product of the transpose and the fact that transposing twice does nothing to A.
$(AA^T)^T = (A^T)^TA^T = AA^T$
$(A+A^T)^T = A^T+(A^T)^T = A^T+A = A+A^T$, where we have used the fact that matrices commute under addition.
Thus, each of these matrices is symmetric as they equal their respective transposes.

(b) **T** **F**: If S is invertible, then $S^{\mathbf{T}}$ is also invertible.
Solution: True. If we can come up with an inverse for S^T, then this will show S^T is invertible. But notice that since S is invertible, $SS^{-1} = S^{-1}S = I_n$ (the $n \times n$ identity matrix), and so we may transpose these two equations to give us: $(SS^{-1})^T = (I_n)^T$ and $(S^{-1}S)^T = (I_n)^T$, but this means $(S^{-1})^TS^T = I_n$ and $S^T(S^{-1})^T = I_n$ (since I_n is symmetric, $(I_n)^T = I_n$), and this demonstrates that the matrix $(S^{-1})^T$ is a left and right (and hence *the*) inverse for S^T, from which we may conclude that S^T is invertible.

(c) **T** **F**: If a row exchange is required to reduce matrix A into upper triangular form U, then A can not be factored as $A = LU$.
Solution: True. If we obtained U by using a row exchange, then in constructing L we must apply the inverse row exchange. However, this will result in an L that is not lower triangular, and hence A is not of the form $A = LU$ with L lower triangular. (However, as mentioned in the textbook, any matrix can be factored as $PA = LU$ for some appropriate exchange matrix P applied to A prior to factorization)

(d) **T** **F**: Suppose A reduces to upper triangular U but U has a 0 in pivot position, then A has no LDU factorization.
Solution: False. If we write $A = LU$, but U contains a 0 pivot, we will be able to place any real number we want in the corresponding diagonal entry to obtain a valid LDU factorization. However, this factorization will not in general be unique.

(B) Solve the following problems showing your work (6 points).

(a) A symmetric matrix S reduces to $\begin{bmatrix} 3 & 9 \\ 0 & 7 \end{bmatrix}$ after performing row operations(except permutations), write the LDU decomposition of S.

Solution: Since S is symmetric and a single row operation reduced it to the above, we see that this operation must have turned a lower-left 9 into a 0. This corresponds to three times the first row. Hence, to undo the process we add three times the first row to obtain:

Answer: $S = \begin{bmatrix} 3 & 9 \\ 9 & 16 \end{bmatrix}$

(b) Let A and B be two symmetric matrices of same size. Which of the followings are symmetric? $A + B$, $A - B$, A^2, AB, ABA, $ABAB$, $ABABA$.

Solution: In general, sums, differences, powers, and palendromes (reads the same forwards as backwards) of symmetric matrices are symmetric. Hence, $A + B$, $A - B$, A^2, ABA, and $ABABA$ are symmetric when A and B are. However, $(AB)^T = B^T A^T = BA$ and $(ABAB)^T = B^T A^T B^T A^T = BABA$, so AB and $ABAB$ are not symmetric in general.

(c) Write the inverse of the permutation matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution: P places row 1 in row 3, row 3 in row 2, and row 2 in row 1. Hence, we need P^{-1} to place row 3 in row 1, row 2 in row 3, and row 1 in row 2. This matrix is:

Answer: $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

10. [10 points] Determine if the set consisting of

(a) all $(x, y, z) \in \mathbb{R}^3$ with $x = y + z$ is a subspace of $\mathbb{R}^3$

(b) all $(x, y, z) \in \mathbb{R}^3$ with $x + z = 2018$ is a subspace of $\mathbb{R}^3$

(c) all 2×2 symmetric matrices is a subspace of M_{22}. (Here M_{22} is the vector space of all 2×2 matrices.)

(d) all polynomials of degree exactly 3 is a subspace of P_5. (Here P_5 is the vector space of all polynomials $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ of degree less than or equal to 5.)

Solution.

(a) Yes.

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} + \begin{bmatrix} c+d \\ c \\ d \end{bmatrix} = \begin{bmatrix} (a+c)+(b+d) \\ (a+c) \\ (b+d) \end{bmatrix},$$

$$r \begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = \begin{bmatrix} ra+rb \\ ra \\ rb \end{bmatrix}, \quad \text{for any } r \in \mathbb{R}.$$

(b) No, because the zero vector does not belong to the given set.

(c) Yes.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} d & f \\ f & g \end{bmatrix} = \begin{bmatrix} a+d & b+f \\ b+f & c+g \end{bmatrix}$$

is a symmetric matrix. Also, for any $r \in \mathbb{R}$

$$r \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} ra & rb \\ rb & rc \end{bmatrix}$$

is a symmetric matrix.

(d) No. All polynomials of degree exactly 3 is not a linear space. For instance, take $q = x^3 + x$ and $p = -x^3$, then $q + p = x$ is a polynomial of degree 1.