MATH 2418: Linear Algebra

Assignment# 4

Due : 02/13, Wednesday　　　　　　　　　　　　　　　　Term : `Spring 2019`

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[**Recommended Problems (do not turn in):** Sec 2.3: 1, 3, 4, 7, 8, 9, 18, 21, 25, 27, 28.
Sec 2.4: 3, 6, 7, 10, 11, 12, 14, 17, 21, 26, 32]

1. Let $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix}$, find the products AB and BA.

Solution:

Number of columns in A = number of rows in B=3, so AB is defined.
Number of columns in B = number of rows in A=3, so BA is defined.

$$AB = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix} = \begin{bmatrix} (1,\ 5,\ 2).(2,\ 0,\ 13) & (1,\ 5,\ 2,).(0,\ 5,\ -2) & (1,\ 5,\ 2).(3,\ 1,\ 0) \\ (1,\ 0,\ 1).(2,\ 0,\ 13) & (1,\ 0,\ 1).(0,\ 5,\ -2) & (1,\ 0,\ 1).(3,\ 1,\ 0) \\ (3,\ -2,\ 4).(2,\ 0,\ 13) & (3,\ -2,\ 4).(0,\ 5,\ -2) & (3,\ -2,\ 4).(3,\ 1,\ 0) \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} (2,\ 0,\ 3).(1,\ 1,\ 3) & (2,\ 0,\ 3).(5,\ 0,\ -2) & (2,\ 0,\ 3).(2,\ 1,\ 4) \\ (0,\ 5,\ 1).(1,\ 1,\ 3) & (0,\ 5,\ 1).(5,\ 0,\ -2) & (0,\ 5,\ 1).(2,\ 1,\ 4) \\ (13,\ -2,\ 0).(1,\ 1,\ 3) & (13,\ -2,\ 0).(5,\ 0,\ -2) & (13,\ -2,\ 0).(2,\ 1,\ 4) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 4 & 16 \\ 8 & -2 & 9 \\ 11 & 65 & 24 \end{bmatrix}$$

2. For the matrices $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 3 \end{bmatrix}$, verify the associative property of the matrix multiplication: $(AB)C = A(BC)$.

Solution: Although showing $A(BC) = (AB)C$ in this case is not a formal proof that associativity always works, it *does* always work. This is in part due to the fact that multiplication of scalars is associative, but is mostly due to the fact that we can think of matrices as **linear transformations** (c.f. Problem 33 of Section 2.1, a recommended problem), that is, as *functions* giving us vectors with vectors as inputs. And since the composition of functions is associative, so too is multiplication of matrices.

First, $AB = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix}$ and $BC = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 14 & 13 \\ 7 & 22 \end{bmatrix}$.

Then $$(AB)C = \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 83 & 122 \\ 6 & 35 \\ -3 & 101 \end{bmatrix} \qquad (1)$$

and $$A(BC) = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 13 \\ 14 & 13 \\ 7 & 22 \end{bmatrix} = \begin{bmatrix} 83 & 122 \\ 6 & 35 \\ -3 & 101 \end{bmatrix} \qquad (2)$$

From (1) and (2), we have $A(BC) = (AB)C$

3. Consider the 3×3 matrix $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$

(a) Perform 3 elementary row operations to reduce the matrix A into an upper triangular matrix U.

Solution:

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 5 & 2 \\ 0 & -5 & -1 \\ 3 & -2 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 5 & 2 \\ 0 & -5 & -1 \\ 0 & -17 & -2 \end{bmatrix} \xrightarrow{R_3 - \frac{17}{5}R_2} \begin{bmatrix} 1 & 5 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & \frac{7}{5} \end{bmatrix} = U$$

(b) Write down the elementary matrices E_{21}, E_{31}, E_{32} corresponding to the row operations in part (a).

Solution:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{17}{5} & 1 \end{bmatrix}$$

(c) Write down a single matrix M such that $MA = U$ is upper triangular.

Solution: Since $E_{32}E_{31}E_{21}A = U$, the single matrix M is given by

$$M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{17}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{17}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{2}{5} & -\frac{17}{5} & 1 \end{bmatrix}.$$

4. Consider the matrix A as in previous question and the vector $\mathbf{b} = (1, 2, 1)$, solve the system $A\mathbf{x} = \mathbf{b}$.

Solution:
Writing down the augmented matrix $[A\ \mathbf{b}]$ and using the matrix $M = E_{32}E_{31}E_{21}$, we have the solution to $A\mathbf{x} = \mathbf{b}$ is the same as to $MA\mathbf{x} = M\mathbf{b}$, hence we will use the augmented matrix $[MA\ M\mathbf{b}]$:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{2}{5} & -\frac{17}{5} & 1 \end{bmatrix} \Rightarrow [MA:\ M\mathbf{b}] = [U:\ M\mathbf{b}] = \begin{bmatrix} 1 & 5 & 2: & 1 \\ 0 & -5 & -1: & 1 \\ 0 & 0 & \frac{7}{5}: & -\frac{27}{5} \end{bmatrix}$$

From the last row of $[U:\ M\mathbf{b}]$ we get $\frac{7}{5}x_3 = -\frac{27}{5} \Rightarrow x_3 = -\frac{27}{7}$, performing back substitution on the row #2 $\Rightarrow -5x_2 - x_3 = 1 \Rightarrow x_2 = \frac{4}{7}$ and, once more, performing back substitution on the first row $\Rightarrow x_1 + 5x_2 + 2x_3 = 1 \Rightarrow x_1 = \frac{41}{7}$, therefore the solution is $\mathbf{x} = \frac{1}{7}(41, 4, -27)$.

5. Given elementary matrices $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, find the following products

(a) $E_1E_2 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,-2) & (1,0,0)\cdot(0,1,0) & (1,0,0)\cdot(0,0,1) \\ (0,1,0)\cdot(1,0,-2) & (0,1,0)\cdot(0,1,0) & (0,1,0)\cdot(0,0,1) \\ (2,0,1)\cdot(1,0,-2) & (2,0,1)\cdot(0,1,0) & (2,0,1)\cdot(0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $E_2E_1 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,2) & (1,0,0)\cdot(0,1,0) & (1,0,0)\cdot(0,0,1) \\ (0,1,0)\cdot(1,0,2) & (0,1,0)\cdot(0,1,0) & (0,1,0)\cdot(0,0,1) \\ (-2,0,1)\cdot(1,0,2) & (-2,0,1)\cdot(0,1,0) & (-2,0,1)\cdot(0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $E_3E_4 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,0) & (1,0,0)\cdot(0,1,0) & (1,0,0)\cdot(0,4,1) \\ (0,1,-4)\cdot(1,0,0) & (0,1,-4)\cdot(0,1,0) & (0,1,-4)\cdot(0,4,1) \\ (0,0,1)\cdot(1,0,0) & (0,0,1)\cdot(0,1,0) & (0,0,1)\cdot(0,4,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) $E_4E_3 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,0) & (1,0,0)\cdot(0,1,0) & (1,0,0)\cdot(0,-4,1) \\ (0,1,4)\cdot(1,0,0) & (0,1,4)\cdot(0,1,0) & (0,1,4)\cdot(0,-4,1) \\ (0,0,1)\cdot(1,0,0) & (0,0,1)\cdot(0,1,0) & (0,0,1)\cdot(0,-4,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Let $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 4 & 6 & 0 \end{bmatrix}$.

(a) For any 4×3 matrix A, what is the column 3 of AB? Explain your answer.
Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$ and rewrite B as column vectors$[C_1\ C_2\ C_3]$ with $C_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Then the column 3 of $AB = AC_3 = \begin{bmatrix} R_1 \cdot C_3 \\ R_2 \cdot C_3 \\ R_3 \cdot C_3 \\ R_4 \cdot C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) For any 3×2 matrix C, what is the row 2 of BC? Explain your answer.
Solution:

Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ and rewrite B as row vectors $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$ with $R_2 = [0\ 0\ 0]$.

Then the row 2 of $BC = R_2C = [R_2 \cdot C_1\ R_2 \cdot C_2] = [0\ 0]$

(c) Is it possible to find a 3×3 matrix D such that $DB = I_3$, the 3×3 identity matrix? Explain.
Solution:

No. By the same token of (a), for any 3×3 matrix D, the column 3 of DB is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, but the 3×3 identity matrix that has column 3 as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) Is it possible to find a 3×3 matrix F such that $BF = I_3$, the 3×3 identity matrix? Explain.
Solution:

No. By the same token of (b), for any 3×3 matrix F, the row 2 of BF is $[0\ 0\ 0]$, but the 3×3 identity matrix that has row 2 as $[0\ 1\ 0]$

7. Let $A = \begin{bmatrix} 1 & 2 & -5 \\ -1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 10 \\ 1 & -2 \\ 2 & 1 \end{bmatrix}$, without calculating AB, A^2 and BA, calculate the following:

(a) the entry $(AB)_{32}$ of AB

Solution:
$(AB)_{32}$=(row 3 of A) $\cdot$ (column 2 of B)=$(3, -2, 4) \cdot (10, -2, 1) = 30 + 4 + 4 = 38$.

(b) the entry $(BA)_{23}$ of BA

Solution:
Because the matrix A is 3×3, matrix B is 3×2, so BA is **undefined**.

(c) row 1 of AB

Solution:

$$\text{row 1 of } AB = [\text{ row 1 of } A]B = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 10 \\ 1 & -2 \\ 2 & 1 \end{bmatrix} = [-11, \quad 1]$$

(d) column 3 of A^2.

Solution:

$$\text{column 3 of } A^2 = A[\text{ column 3 of } A] = \begin{bmatrix} 1 & 2 & -5 \\ -1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -23 \\ 9 \\ -1 \end{bmatrix}$$

8. Compute the following products:

(a) $$\begin{bmatrix} a & 4 & 1 \\ 0 & b & 5 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} d & 1 & 2 \\ 0 & e & 2 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} ad & a+4e & 2a+8+f \\ 0 & be & 2b+5f \\ 0 & 0 & cf \end{bmatrix}$$

(b) $$\begin{bmatrix} a & 0 & 0 \\ 2 & b & 0 \\ 3 & 3 & c \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 1 & e & 0 \\ 2 & 1 & f \end{bmatrix} = \begin{bmatrix} ad & 0 & 0 \\ 2d+b & be & 0 \\ 3d+3+2c & 3e+c & cf \end{bmatrix}$$

(c) $$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 3 & 5 & -7 \\ 2 & 4 & 1 \\ -9 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3a & 5a & -7a \\ 2b & 4b & b \\ -9c & 2c & 6c \end{bmatrix}$$

(d) $$\begin{bmatrix} 3 & 5 & -7 \\ 2 & 4 & 1 \\ -9 & 2 & 6 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 3a & 5b & -7c \\ 2a & 4b & c \\ -9a & 2b & 6c \end{bmatrix}$$

(e) $$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix}$, find the product AB as a sum of columns of A times rows of B.

Solution:

$$AB = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 13 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 3 \\ 2 & 0 & 3 \\ 6 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 25 & 5 \\ 0 & 0 & 0 \\ 0 & -10 & -2 \end{bmatrix} + \begin{bmatrix} 26 & -4 & 0 \\ 13 & -2 & 0 \\ 52 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix}$$

10. Answer the following(you DO NOT need to show your work).

(a) Write down two 2×2 non-zero matrices A and B such that $AB = 0$, the 2×2 zero matrix.

Answer: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) Given $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ write down the elementary matrices E_1 and E_2 such that

$$E_1 A = \begin{bmatrix} 3 & 4 \\ 7 & 10 \end{bmatrix} \text{ and } E_2 A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer: $E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) Given $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ write down the matrices D_1 and D_2 such that

$$D_1 A = \begin{bmatrix} 30 & 40 \\ -3 & -6 \end{bmatrix} \text{ and } AD_2 = \begin{bmatrix} 30 & -12 \\ 10 & -6 \end{bmatrix}$$

Answer: $D_1 = \begin{bmatrix} 10 & 0 \\ 0 & -3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 10 & 0 \\ 0 & -3 \end{bmatrix}$

(d) For any two 2×2 matrices A and B, which of the following are guaranteed to be equal to $(A + B)^2$?

(a) $A^2 + 2AB + B^2$ (b) $A^2 + AB + BA + B^2$ (c) $A^2 + B^2$ (d) $A^2 - 2AB + B^2$
Answer: $(A + B)^2 = A^2 + AB + BA + B^2$ (Note: $AB \neq BA$ in general.)

(e) For any two 2×2 matrices A and B, which of the following are guaranteed to be equal to $(A+B)(A-B)$?

(a) $A^2 + B^2$ (b) $A^2 - AB + BA - B^2$ (c) $A^2 - B^2$ (d) $A^2 - 2AB + B^2$
Answer: $(A + B)(A - B) = A^2 - AB + BA - B^2$