MATH 2418: Linear Algebra

Assignment# 3

Due : 02/06, Wednesday Term `Spring, 2019`

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Recommended Text Book Problems (do not turn in): [Sec 2.1: 9, 10, 15, 16, 17, 19, 26, 29, 31, 33; Sec 2.2: 5, 6, 7, 12, 14, 15, 19, 23, 24, 25]

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1. Solve the linear system $\begin{cases} x & = 1 \\ x + y & = 2 \\ z & = 2 \end{cases}$. Also, draw the column picture.

Solution:

Elimination leads to the following,

$$\begin{cases} x & = 1 \\ y & = 1 \\ z & = 2 \end{cases}$$

which gives the solution, $(x, y, z) = (1, 1, 2)$.
The system can be represented as $A\mathbf{x} = \mathbf{b}$, where,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

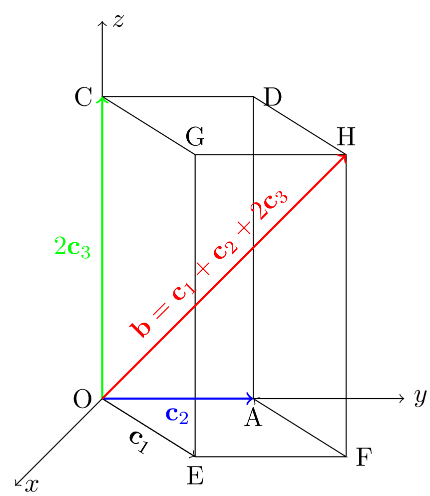
We have three column vectors from A, with $A = [\mathbf{c}_1|\mathbf{c}_2|\mathbf{c}_3]$ where,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = (x, y, z) = (1, 1, 2)$, we can write $\mathbf{b}$ as a linear combination of columns of A as:

$$1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Hence, the column picture is the following:



Column Picture.

2. Given linear system $\begin{cases} x + y = 6 \\ 3x - y = 2 \end{cases}$

   (a) Write down the corresponding matrix equation $A\mathbf{x} = \mathbf{b}$.
   (b) Solve the system.
   (c) Draw the row picture and the column picture.
   (d) Write $\mathbf{b}$ as a linear combination of columns of A.

Solution:

(a) The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

(b) First note, that summing two equations together gives us $4x = 8$, thus $x = 2$. By plugging it in either equation we find $y = 4$.

(c) Row picture:

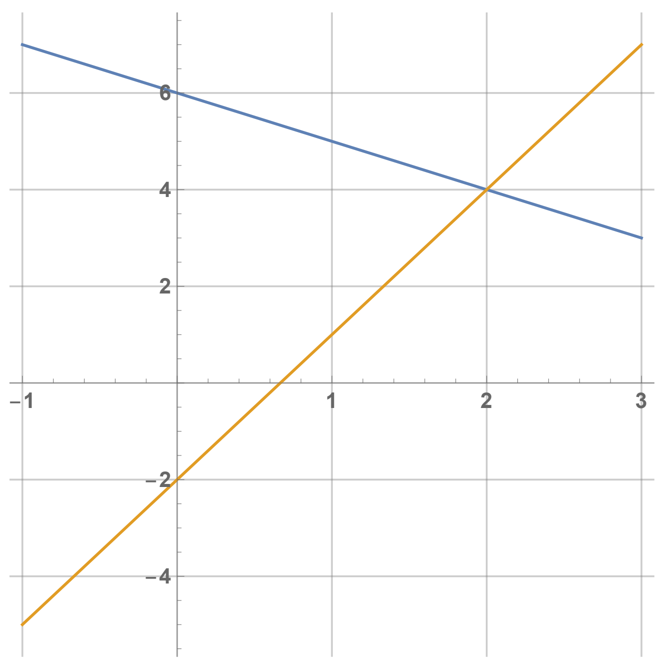


Figure 1: Row picture. Blue and orange lines are graphs of the first and the second equation, respectively.

Column picture:

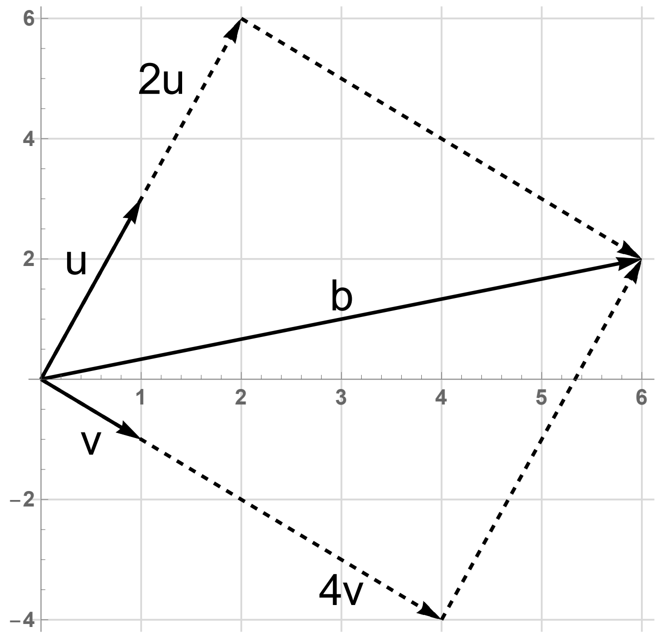


Figure 2: Column picture. Here $\mathbf{u} = (1,3)$, $\mathbf{v} = (1,-1), \mathbf{b} = (6,2)$
.

(d)

$$\mathbf{b} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

3. Consider the transformation $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\mathrm{T}(x, y, z) = (x + y + 3z,\ x - 2y - z)$ (i.e. T is a function that maps every vector $(x, y, z) \in \mathbb{R}^3$ to a vector $(x+y+3z,\ x-2y-z) \in \mathbb{R}^2$). Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be any two vectors in $\mathbb{R}^3$ and 'c' be any scalar. Prove that

   (a) $\mathrm{T}(\mathbf{u} + \mathbf{v}) = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$

   (b) $\mathrm{T}(c\mathbf{u}) = c\mathrm{T}(\mathbf{u})$

   Solution:

   (a) Given $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, so $\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$

$$\begin{aligned}
\mathrm{T}(\mathbf{u} + \mathbf{v}) &= \mathrm{T}(u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3) \\
&= ((u_1 + v_1) + (u_2 + v_2) + 3(u_3 + v_3),\ (u_1 + v_1) - 2(u_2 + v_2) - (u_3 + v_3)) \\
&= (u_1 + u_2 + 3u_3 + v_1 + v_2 + 3v_3,\ u_1 - 2u_2 - u_3 + v_1 - 2v_2 - v_3) \\
&= (u_1 + u_2 + 3u_3,\ u_1 - 2u_2 - u_3) + (v_1 + v_2 + 3v_3,\ v_1 - 2v_2 - v_3) \\
&= \mathrm{T}(u_1, u_2, u_3) + \mathrm{T}(v_1, v_2, v_3) \\
&= \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})
\end{aligned}$$

   (b) $c\mathbf{u} = c(u_1, u_2, u_3) = (cu_1, cu_2, cu_3)$

$$\begin{aligned}
\mathrm{T}(c\mathbf{u}) &= \mathrm{T}(cu_1, cu_2, cu_3) \\
&= (cu_1 + cu_2 + 3(cu_3),\ cu_1 - 2(cu_2) - cu_3) \\
&= (c(u_1 + u_2 + 3u_3),\ c(u_1 - 2u_2 - u_3)) \\
&= c\left((u_1 + u_2 + 3u_3),\ (u_1 - 2u_2 - u_3)\right) \\
&= c\mathrm{T}(u_1, u_2, u_3) \\
&= c\mathrm{T}(\mathbf{u})
\end{aligned}$$

4. Consider the transformation T defined in Q.N#3:

(a) Find the 2×3 matrix A such that $\mathrm{T}(\mathbf{u}) = A\mathbf{u}$, where $\mathbf{u}$ is any vector in $\mathbb{R}^3$.

Solution:
We have $\mathrm{T}(\mathbf{u}) = \mathrm{T}(u_1, u_2, u_3) = (u_1 + u_2 + 3u_3,\ u_1 - 2u_2 - u_3)$

i.e. $\mathrm{T}(\mathbf{u}) = \mathrm{T}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 + 3u_3 \\ u_1 - 2u_2 - u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A\mathbf{u}$

Therefore $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}$

(b) Let $\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, find $\mathrm{T}(\mathbf{i}),\ \mathrm{T}(\mathbf{j})$, and $\mathrm{T}(\mathbf{k})$.

Solution:

$$\begin{aligned} \mathrm{T}(\mathbf{i}) &= \mathrm{T}(1, 0, 0) = (1 + 0 + 3 \cdot 0, 1 - 2 \cdot 0 - 0) = (1, 1) \\ \mathrm{T}(\mathbf{j}) &= \mathrm{T}(0, 1, 0) = (0 + 1 + 3 \cdot 0, 0 - 2 \cdot 1 - 0) = (1, -2) \\ \mathrm{T}(\mathbf{k}) &= \mathrm{T}(0, 0, 1) = (0 + 0 + 3 \cdot 1, 0 - 2 \cdot 0 - 1) = (3, -1) \end{aligned}$$

(c) Write the 2×3 matrix P whose columns are the vectors $\mathrm{T}(\mathbf{i}),\ \mathrm{T}(\mathbf{j})$, and $\mathrm{T}(\mathbf{k})$ from part (b).

Solution: $P = \begin{bmatrix} \mathrm{T}(\mathbf{i}) & \mathrm{T}(\mathbf{j}) & \mathrm{T}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}$

5. Find the matrix for the following linear transformations:

(a) Rotation of vectors in $\mathbb{R}^2$ by 120° clockwise.

Solution: The general rotation matrix is: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, here the angle $\theta = -120°$ or $-2\pi/3$ so the matrix is $\begin{bmatrix} \cos\left(-\frac{2\pi}{3}\right) & -\sin\left(-\frac{2\pi}{3}\right) \\ \sin\left(-\frac{2\pi}{3}\right) & \cos\left(-\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

(b) Maps any vector (x, y) in $\mathbb{R}^2$ to the vector $(2x + y,\ 3x - 4y)$ in $\mathbb{R}^2$.

Solution: Since $\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3x - 4y \end{bmatrix}$, the matrix is $\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$

(c) Maps any vector (x, y) in $\mathbb{R}^2$ to give the vector $(x + y,\ 2y,\ 2x - 3y)$ in $\mathbb{R}^3$

Solution: Since $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2y \\ 2x - 3y \end{bmatrix}$, the matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & -3 \end{bmatrix}$

6. Given linear system $\begin{cases} 2x + 4y - 3z = 7 \\ 8x + 17y - 3z = 39 \\ -3x - 3y + 7z = -2 \end{cases}$.

(a) Solve by reducing into upper triangular form and back substitution.

Solution:

$$\begin{cases} \mathbf{2}x + 4y - 3z = 7 \\ 8x + 17y - 3z = 39 \\ -3x - 3y + 7z = -2 \end{cases} \xrightarrow{Eq(2)-4\cdot Eq(1)} \begin{cases} \mathbf{2}x + 4y - 3z = 7 \\ y + 9z = 11 \\ -3x - 3y + 7z = -2 \end{cases} \xrightarrow{Eq(3)+\frac{3}{2}Eq(1)} \begin{cases} 2x + 4y - 3z = 7 \\ \mathbf{1}y + 9z = 11 \\ 3y + \frac{5}{2}z = \frac{17}{2} \end{cases}$$

$$\xrightarrow{Eq(3)-3\cdot Eq(2)} \begin{cases} 2x + 4y - 3z = 7 \\ 1y + 9z = 11 \\ -\frac{49}{2}z = -\frac{49}{2} \end{cases}$$

Back substitution:
From the last equation, we have $z = 1$.
From $y + 9z = 11$, we have $y = 11 - 9z$, so $y = 2$.
From $2x + 4y - 3z = 7$, we have $x = \frac{7-4y+3z}{2}$, so $x = 1$.

Finally, the solution is $\begin{cases} x = 1 \\ y = 2 \\ z = 1 \end{cases}$.

(b) List all multipliers used and circle all the pivots.

Solution:

The pivots are circled. The multipliers are: $l_{21} = 4,\ l_{31} = -\frac{3}{2},\ l_{32} = 3$

7. Consider the linear system $\begin{cases} ax - 3y = 6 \\ 3x + 6y = -12 \end{cases}$

   (a) For what value(s) of 'a' does the elimination break down (i) permanently (ii) temporarily?

   (b) Solve the system after fixing the temporary break down.

   (c) Solve the system in case of permanent break down.

Solution:

(a) (i) Let $a = -\frac{3}{2}$:

$$\begin{cases} -\frac{3}{2}x - 3y = 6 \\ 3x + 6y = -12 \end{cases} \tag{1}$$

Now there is really only one equation $3x + 6y = 12$ as it is a multiple of the first equation. If we apply elimination, the system reduces to

$$\begin{cases} -\frac{3}{2}x - 3y = 6 \\ 0x + 0y = 0 \end{cases}$$

and there is now way we can obtain 2 pivots, hence, elimination breaks down permanently.

(ii) Let $a = 0$:

$$\begin{cases} 0x - 3y = 6 \\ 3x + 6y = -12 \end{cases} \tag{2}$$

There is 0 in a pivot position but a row exchange will give us pivots on each pivot position:

$$\begin{cases} 3x + 6y = -12 \\ \qquad -3y = 6 \end{cases}$$

So the elimination fails temporarily if $a = 0$ The new system is already triangular.

(b) System (2) has exactly one solution:

$$\begin{cases} x = 0 \\ y = -2 \end{cases}$$

i.e. $(x, y) = (0, -2)$

(c) System (1) has infinitely many solutions $(x, y) = (-4 - 2t, t); t \in \mathbb{R}$.

8. Is the system $\begin{cases} 3x + \ \ 4y - 2z = 15 \\ 9x + 13y - 3z = 26 \\ -3x - \ \ 3y + 5z = 12 \end{cases}$ solvable? If not, change the number 15 on the right side of the first equation so that the system has a solution.

Solution:

$$\begin{cases} 3x + \ \ 4y - 2z = 15 \\ 9x + 13y - 3z = 26 \\ -3x - \ \ 3y + 5z = 12 \end{cases} \xrightarrow[Eq(2)-3\cdot Eq(1)]{Eq(3)+Eq(1)} \begin{cases} 3x + 4y - 2z = 15 \\ \qquad y + 3z = -19 \\ \qquad y + 3z = 27 \end{cases} \xrightarrow{Eq(3)-Eq(2)} \begin{cases} 3x + 4y - 2z = 15 \\ \qquad y + 3z = -19 \\ \quad 0y + 0z = 46 \end{cases}$$

which is not possible (because the last equation says $0 = 46$) . Then the system is not solvable.

If we change the number 15 to be b we will get:

$$\begin{cases} 3x + \ \ 4y - 2z = b \\ 9x + 13y - 3z = 26 \\ -3x - \ \ 3y + 5z = 12 \end{cases} \xrightarrow[Eq(2)-3\cdot Eq(1)]{Eq(3)+Eq(1)} \begin{cases} 3x + 4y - 2z = b \\ \qquad y + 3z = 26 - 3b \\ \qquad y + 3z = 12 + b \end{cases} \xrightarrow{Eq(3)-Eq(2)} \begin{cases} 3x + 4y - 2z = b \\ \qquad y + 3z = 26 - 3b \\ \quad 0y + 0z = 4b - 14 \end{cases}$$

So the system will be solvable if $4b - 14 = 0 \implies b = \frac{7}{2}$.

9. Reduce the matrix $A = \begin{bmatrix} a & 2 & 3 \\ a & a+1 & 4 \\ a & a+1 & a \end{bmatrix}$ into upper triangular form. Determine the three values of 'a' for which the elimination fails to give three pivots.

Solution: Reducing A to upper triangular form:

$$\begin{bmatrix} a & 2 & 3 \\ a & a+1 & 4 \\ a & a+1 & a \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - R_1} \begin{bmatrix} a & 2 & 3 \\ 0 & a-1 & 1 \\ a & a+1 & a \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - R_1} \begin{bmatrix} a & 2 & 3 \\ 0 & a-1 & 1 \\ 0 & a-1 & a-3 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - R_2} \begin{bmatrix} a & 2 & 3 \\ 0 & a-1 & 1 \\ 0 & 0 & a-4 \end{bmatrix}$$

For $a = 0, 1, 4$, the elimination fails to give three pivots.

10. True or False. Circle your answer.

(a) **T** **F**: Consider the augmented matrix $[A \quad : \mathbf{b}] = \begin{bmatrix} 4 & 5 & : b_1 \\ 0 & r & : b_2 \end{bmatrix}$. The existence and uniqueness of the solution depend on all of r, b_1, b_2.
Solution:
False! it depends on r and b_2 only.

(b) **T** **F**: If elimination fails permanently, the linear system has no solution.
Solution:
False! it may have infinitely many solutions.

(c) **T** **F**: There is only one possible linear system which reduces to $\begin{cases} x + \quad y = 1 \\ \qquad 2y = 3 \end{cases}$ after one elimination step.
Solution:
False!.

(d) **T** **F**: If the equations in a linear system are multiplied by scalars, the row picture and the column picture both change.
Solution:
False! only the column picture will change.

(e) **T** **F**: The matrix that projects the vector (x, y) onto $x-$axis to produce $(x, 0)$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
Solution:
True!