MATH 2418: Linear Algebra

Assignment# 2

Due : 01/30, Wednesday Term Spring 2019

[Last Name] [First Name] [Net ID]

Recommended Text Book Problems (do not turn in): [Sec 1.2: # 1, 2, 3, 4, 7, 8, 12, 13, 17, 31
Sec 1.3: 1, 2, 3, 5, 8, 9, 14]

1. Find all real values of 'm' so that angle between the vectors $\mathbf{u} = (m+1, -m+2, -3)$ and $\mathbf{v} = (-3, m+1, -m+2)$ is $120°$.

Solution: Recall that the cosine of the angle θ between any two non-zero vectors $\mathbf{u}, \mathbf{v}$ can be found by the formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}.$$

In our case $\theta = 120°$, thus $\cos\theta = -\frac{1}{2}$. Therefore, we need do find all real values of m which satisfy the equation

$$\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|} = -\frac{1}{2}.$$

It is easy to see that

$$\mathbf{u}\cdot\mathbf{v} = (m+1)(-3) + (-m+2)(m+1) + (-3)(-m+2) = -7 + m - m^2$$

and

$$\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{(m+1)^2 + (-m+2)^2 + (-3)^2} = \sqrt{2(7 - m + m^2)}.$$

Note that, for any real value of m we have

$$\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|} = \frac{-7 + m - m^2}{2(7 - m + m^2)} = -\frac{1}{2}.$$

Hence, the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is $120°$ for any real value of m.

2. Given vectors $\mathbf{u} = (1, 2, -3)$ and $\mathbf{v} = (-3, 1, 2)$ in $\mathbb{R}^3$:

   (a) Calculate the dot product: $\mathbf{u} \cdot \mathbf{v}$

   Solution:

$$\mathbf{u} \cdot \mathbf{v} = (1, 2, -3) \cdot (-3, 1, 2) = 1 \times (-3) + 2 \times 1 + (-3) \times 2 = -3 + 2 - 6 = -7$$

   (b) Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$

   Solution:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$
$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 1^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

   (c) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$

   Solution:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-7}{\sqrt{14}\sqrt{14}} = -\frac{7}{14} = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

   (d) Find the unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$.

   Solution:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \tfrac{1}{\sqrt{14}}(1, 2, -3) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$$

   (e) Write a vector $\mathbf{a}$ of length 3 that is in the opposite direction of $\mathbf{u}$.

   Solution:

$$\mathbf{a} = -3\hat{\mathbf{u}} = \tfrac{-3}{\sqrt{14}}(1, 2, -3) = \left(\frac{-3}{\sqrt{14}}, \frac{-6}{\sqrt{14}}, \frac{9}{\sqrt{14}}\right)$$

3. Let α, β, γ be the angles made by a vector (or a line) with positive x, y, and $z-$axis respectively. Then the numbers

$$l = \cos\alpha,\ \ m = \cos\beta,\ \ n = \cos\gamma$$

are called the **direction cosines** of the the vector (or the line).

(a) Find the direction cosines l, m, n of the vector $\mathbf{u} = (1, 2, 3)$

Solution:

$$l = \cos(\alpha) = \frac{\mathbf{u}\cdot\mathbf{i}}{\|\mathbf{u}\|\|\mathbf{i}\|} = \frac{(1,2,3)\cdot(1,0,0)}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}} = \frac{\sqrt{14}}{14}$$

$$m = \cos(\beta) = \frac{\mathbf{u}\cdot\mathbf{j}}{\|\mathbf{u}\|\|\mathbf{j}\|} = \frac{(1,2,3)\cdot(0,1,0)}{\sqrt{1^2+2^2+3^2}} = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$

$$n = \cos(\gamma) = \frac{\mathbf{u}\cdot\mathbf{k}}{\|\mathbf{u}\|\|\mathbf{k}\|} = \frac{(1,2,3)\cdot(0,0,1)}{\sqrt{1^2+2^2+3^2}} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

(b) Find the direction cosines l, m, n of the vector $\mathbf{u} = (a, b, c)$.

Solution:

$$l = \cos(\alpha) = \frac{\mathbf{u}\cdot\mathbf{i}}{\|\mathbf{u}\|\|\mathbf{i}\|} = \frac{(a,b,c)\cdot(1,0,0)}{\sqrt{a^2+b^2+c^2}} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$m = \cos(\beta) = \frac{\mathbf{u}\cdot\mathbf{j}}{\|\mathbf{u}\|\|\mathbf{j}\|} = \frac{(a,b,c)\cdot(0,1,0)}{\sqrt{a^2+b^2+c^2}} = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \cos(\gamma) = \frac{\mathbf{u}\cdot\mathbf{k}}{\|\mathbf{u}\|\|\mathbf{k}\|} = \frac{(a,b,c)\cdot(0,0,1)}{\sqrt{a^2+b^2+c^2}} = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

4. (a) Use the triangle inequality: $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ to prove that

(i) $\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

(ii) $\|\mathbf{u}\| - \|\mathbf{v}\| \le \|\mathbf{u}-\mathbf{v}\|$

Solution:

(i) Use $-\mathbf{v}$ to replace $\mathbf{v}$ in the triangle inequality, we have:

$$\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+(-\mathbf{v})\| \le \|\mathbf{u}\| + \|-\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$$

Thus

$$\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

(ii) Notice that $\|\mathbf{u}\| = \|(\mathbf{u}-\mathbf{v})+\mathbf{v}\|$, by applying the triangle inequality, we have:

$$\|\mathbf{u}\| = \|(\mathbf{u}-\mathbf{v})+\mathbf{v}\| \le \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$$

Subtracting $\|\mathbf{v}\|$ from both sides:

$$\|\mathbf{u}\| - \|\mathbf{v}\| \le \|\mathbf{u}-\mathbf{v}\|$$

(b) If $\|\mathbf{u}\| = 19$ and $\|\mathbf{v}\| = 24$, what are the smallest and largest possible values of $\|\mathbf{u}-\mathbf{v}\|$ and $\|\mathbf{v}-\mathbf{u}\|$?

Solution:
From the result above,
$\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| = 19 + 24 = 43$, so the largest possible value is 43.
$\|\mathbf{v}-\mathbf{u}\| \ge \|\mathbf{v}\| - \|\mathbf{u}\| = 24 - 19 = 5$, so the smallest possible value is 5.

5. (a) Given any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, determine the scalar 'c' so that the vector $\mathbf{u} - c\mathbf{v}$ is perpendicular to $\mathbf{v}$.

Solution:

Let $\mathbf{u} \perp (\mathbf{u} - c\mathbf{v})$, then
$(\mathbf{u} - c\mathbf{v}) \cdot \mathbf{v} = 0$
$\implies \mathbf{u} \cdot \mathbf{v} - c\mathbf{v} \cdot \mathbf{v} = 0$
$\implies c\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}$
$\implies c\|\mathbf{v}\|^2 = \mathbf{u} \cdot \mathbf{v}$
$\implies c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$

(b) Let $\mathbf{v} = (4, 1, 3)$ and $\mathbf{u} = (1, 1, 1)$, use part (a) to find a non zero vector that is perpendicular to $\mathbf{v}$.

Solution:

We can choose the vector to be $\mathbf{u} - c\mathbf{v}$.

From part (a), $c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} = \frac{(1,1,1)\cdot(4,1,3)}{4^2+1^2+3^2} = \frac{4+1+3}{26} = \frac{8}{26} = \frac{4}{13}$.

So the vector $\mathbf{u} - c\mathbf{v} = (1, 1, 1) - \frac{4}{13}(4, 1, 3) = \left(-\frac{3}{13}, \frac{9}{13}, \frac{1}{13}\right)$ is perpendicular to $\mathbf{v} = (4, 1, 3)$.

6. Given the 3×3 matrix $A = \begin{bmatrix} -3 & 2 & -3 \\ 2 & 3 & -8 \\ 3 & -2 & 3 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$, calculate $A\mathbf{x}$

(a) as a linear combination of columns of A

Solution:

$$A\mathbf{x} = \begin{bmatrix} -3 & 2 & -3 \\ 2 & 3 & -8 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix} = \begin{bmatrix} -9+16-3 \\ 6+24-8 \\ 9-16+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 22 \\ -4 \end{bmatrix}$$

(b) with entries as dot products of rows of A and vector $\mathbf{x}$.

Solution:

$$A\mathbf{x} = \begin{bmatrix} -3 & 2 & -3 \\ 2 & 3 & -8 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} (-3,2,-3)\cdot(3,8,1) \\ (2,3,-8)\cdot(3,8,1) \\ (3,-2,3)\cdot(3,8,1) \end{bmatrix} = \begin{bmatrix} -9+16-3 \\ 6+24-8 \\ 9-16+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 22 \\ -4 \end{bmatrix}$$

7. Let matrix $A = \begin{array}{c} \\ S1 \\ S2 \\ S3 \end{array} \begin{array}{c} \begin{array}{ccc} E1 & E2 & E3 \end{array} \\ \begin{bmatrix} 70 & 80 & 90 \\ 90 & 90 & 80 \\ 50 & 70 & 100 \end{bmatrix} \end{array}$ represent the Exam 1(E1), Exam 2(E2), and Exam 3(E3) scores (out of 100 points each) of 3 students $S_1,\ S_2$, and S_3. The vector $\mathbf{w} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ represents the Exam 1, Exam 2, and Exam 3 weights ($20\%, 30\%$, and 50% respectively). Calculate and explain the meaning of $A\mathbf{w}$.

Solution:

$$A\mathbf{w} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \end{array} \begin{array}{c} \begin{array}{ccc} E1 & E2 & E3 \end{array} \\ \begin{bmatrix} 70 & 80 & 90 \\ 90 & 90 & 80 \\ 50 & 70 & 100 \end{bmatrix} \end{array} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \end{array} \begin{array}{c} \text{Total scores} \\ \begin{bmatrix} 70 \times 0.2 + 80 \times 0.3 + 90 \times 0.5 \\ 90 \times 0.2 + 90 \times 0.3 + 80 \times 0.5 \\ 50 \times 0.2 + 70 \times 0.3 + 100 \times 0.5 \end{bmatrix} \end{array} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \end{array} \begin{array}{c} \text{Total scores} \\ \begin{bmatrix} 83 \\ 85 \\ 81 \end{bmatrix} \end{array}$$

$A\mathbf{w}$ represents weighted average of each student in all the three exams.

8. Given $A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(a) Write the linear system corresponding to $A\mathbf{x} = \mathbf{b}$.

Solution: The corresponding linear system is: $\begin{cases} -x_1 + 3x_2 \phantom{{}+x_3} = b_1 \\ 2x_1 + x_2 + x_3 = b_2 \\ x_2 + x_3 = b_3 \end{cases}$

(b) Solve the linear system.

Solution: Rewriting the linear system:

$$\begin{aligned} -x_1 + 3x_2 &= b_1 && (1) \\ 2x_1 + x_2 + x_3 &= b_2 && (2) \\ x_2 + x_3 &= b_3 && (3) \end{aligned}$$

Equation (2) - Equation (3) gives:

$$\begin{aligned} 2x_1 &= b_2 - b_3 && (4) \\ x_1 &= \frac{1}{2}(b_2 - b_3) && (5) \end{aligned}$$

Solving Equation (1) for x_2:

$$\begin{aligned} x_2 &= \frac{1}{3}(b_1 + x_1) && (6) \\ &= \frac{b_1}{3} + \frac{1}{6}(b_2 - b_3) && (7) \end{aligned}$$

Substituting (6) in (3):

$$\begin{aligned} \frac{b_1}{3} + \frac{1}{6}(b_2 - b_3) + x_3 = b_3&, \\ x_3 &= -\frac{b_1}{3} - \frac{1}{6}(b_2 - b_3) + b_3, \\ x_3 &= \tfrac{1}{6}(-2b_1 - b_2 + 7b_3). && (8) \end{aligned}$$

Thus, we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3b_2 - 3b_3 \\ 2b_1 + b_2 - b_3 \\ -2b_1 - b_2 + 7b_3 \end{bmatrix}. \qquad (9)$$

(c) Write your answer in the form of $\mathbf{x} = A^{-1}\mathbf{b}$. What is A^{-1}?

Solution: From Equation (9) we have,

$$\mathbf{x} = \frac{1}{6} \begin{bmatrix} 0 & 3 & -3 \\ 2 & 1 & -1 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = A^{-1}\mathbf{b}, \text{ where } A^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 3 & -3 \\ 2 & 1 & -1 \\ -2 & 1 & 7 \end{bmatrix}$$

9. (a) Prove that the vectors $\mathbf{u} = (-1, 2, 0)$, $\mathbf{v} = (3, 1, 1)$, $\mathbf{w} = (0, 1, 1)$ are linearly independent.

Solution: Suppose $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, the zero vector. This implies

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -x_1 + 3x_2 \\ 2x_1 + x_2 + x_3 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding linear system is $\begin{cases} -x_1 + 3x_2 \qquad\quad = 0 & (1) \\ 2x_1 + \;\; x_2 + x_3 = 0 & (2) \\ \qquad\quad x_2 + x_3 = 0 & (3) \end{cases}$

Eqn(2) - Eqn (3) $\implies$ $2x_1 = 0 \implies x_1 = 0$. Plugging in $x_1 = 0$ in Eqn(1), we get $x_2 = 0$, and then plugging in $x_2 = 0$ in Eqn(3), we get $x_3 = 0$. So $x_1 = x_2 = x_3 = 0$. Therefore the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

(b) Prove that the vectors $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (3, 1, 1)$, $\mathbf{w} = (5, 5, 3)$ are linearly dependent.

Solution: Suppose $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, the zero vector. This implies

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x_1 + 3x_2 + 5x_3 \\ 2x_1 + x_2 + 5x_3 \\ x_1 + x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve the linear system:

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 \qquad (1) \\ 2x_1 + x_2 + 5x_3 &= 0 \qquad (2) \\ x_1 + x_2 + 3x_3 &= 0 \qquad (3) \end{aligned}$$

Eqn(2)$-$Eqn(3) gives

$$x_1 + 2x_3 = 0 \qquad (4)$$

3*Eqn(2)$-$Eqn(1) gives

$$x_1 + 2x_3 = 0 \qquad (5)$$

Notice that (4) and (5) are the same equation, let $x_3 = t$ where t any real number. Thus from (4) $x_1 = -2x_3 = -2t$ and from (3) $x_2 = -t$. Then $(x_1, x_2, x_3) = (-2t, -t, t)$ are infinitely many solutions, meaning that x_1, x_2, x_3 need not be all 0 (for example: $x_1 = -2, x_2 = -1, x_3 = 1$ is a solution). Therefore the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent.

10. True or False. Circle your answer. **Comment**: With true-false questions in which justification is needed, if we choose "false" we must provide a counter-example to the statement in question. If we choose true, then we must provide a reason why the statement is *always* true.

   (a) **T F**: If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then the set $\{a\mathbf{u}, b\mathbf{v}, c\mathbf{w}\}$ is also linearly independent, where a, b, c are any non zero real numbers.

   Solution: True. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then the equation $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = 0$ for α, β, and γ real numbers, has only $\alpha = \beta = \gamma = 0$ as its solution. Looking at the set $\{a\mathbf{u}, b\mathbf{v}, c\mathbf{w}\}$, let α, β, γ be arbitrary real numbers and let $\alpha' = \alpha a$, $\beta' = \beta b$, $\gamma' = \gamma c$. Then $\alpha'\mathbf{u} + \beta'\mathbf{v} + \gamma'\mathbf{w} = 0$ only when $\alpha' = \beta' = \gamma' = 0$, but this can only occur when $\alpha = \beta = \gamma = 0$, since a, b, and c are *not* 0. This means that $\alpha(a\mathbf{u}) + \beta(b\mathbf{v}) + \gamma(c\mathbf{w}) = 0$ only when $\alpha = \beta = \gamma = 0$, so $\{a\mathbf{u}, b\mathbf{v}, c\mathbf{w}\}$ is a linearly independent set of vectors.

   (b) **T F**: Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three non zero vectors in $\mathbb{R}^2$ such that $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ both, then $\mathbf{v}$ and $\mathbf{w}$ must be parallel to each other.

   Solution: True. Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and $\mathbf{w} = (w_1, w_2)$. Since neither is the zero vector, if $\mathbf{v}$ or $\mathbf{w}$ contains a zero entry (say, for example, $v_1 = 0$) then we must have $u_2v_2 = 0$ which means $u_2 = 0$ since $v_2 \neq 0$ (as $\mathbf{v}$ is not the zero vector). But this means that $w_1 = 0$ too by the reverse reasoning, so $\mathbf{w}$ and $\mathbf{v}$ are parallel. Now, if $\mathbf{u}$ has a 0 entry, then the same reasoning as the above will mean that $\mathbf{v}$ and $\mathbf{w}$ are parallel.

   Finally, assume $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ contain no 0 components. Since $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$, we have $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = 0$ and $\mathbf{u} \cdot \mathbf{w} = u_1w_1 + u_2w_2 = 0$. Solving the first equation for v_1 and the second for w_1 yields $v_1 = \frac{-u_2v_2}{u_1}$ and $w_1 = \frac{-u_2w_2}{u_1}$. Then take their ratio: $\frac{v_1}{w_1} = \frac{-u_2v_2}{u_1}\frac{u_1}{-u_2w_2} = \frac{v_2}{w_2} = c$, so $v_1 = cw_1$ and $v_2 = cw_2$, from which it follows that $\mathbf{v}$ is parallel to $\mathbf{w}$.

   (c) **T F**: Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three non zero vectors in $\mathbb{R}^3$ such that $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ both, then $\mathbf{v}$ and $\mathbf{w}$ must be parallel to each other.

   Solution: False. Consider the vectors $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (0, 0, 1)$. Then $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and the $\mathbf{w}$, yet $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

   (d) **T F**: For fixed length vectors, $\mathbf{u}$ and $\mathbf{v}$, the value of $\mathbf{u} \cdot \mathbf{v}$ is minimum when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other.

   Solution: False. $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. When $\theta = \frac{\pi}{2}$ radians, the dot product is 0. When $\theta = \pi$, the dot product is negative (provided $\mathbf{u}$ and $\mathbf{v}$ are not the zero vector), and hence for such $\mathbf{u}$ and $\mathbf{v}$, we can achieve a smaller value of the dot product than when the vectors are perpendicular.

   (e) **T F**: For fixed length vectors $\mathbf{u}$ and $\mathbf{v}$, the value of $\mathbf{u} \cdot \mathbf{v}$ is maximum when $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

   Solution: True. As in the previous problem, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$. For fixed-length vectors $\mathbf{u}$ and $\mathbf{v}$, this dot product achieves its maximum when $\cos(\theta)$ is maximized. This occurs when $\theta = 0$, meaning that the vectors are in the same direction.

   Note: "Having the same direction" is even more strict than "parallel" as the above reasoning shows. Indeed, two vectors of fixed length will achieve the minimum *or* the maximum dot product for vectors of those lengths precisely when the vectors are pointed in the opposite direction or the same direction, respectively; in both cases, the pairs of vectors are parallel to each other.