MATH 2418: Linear Algebra

Assignment# 1

Due : Wednesday, 01/23/2019 Term :Spring 2019

[Last Name] [First Name] [Net ID] [Lab Section]

Recommended Text Book Problems (do not turn in): [Sec 1.1: # 1, 2, 3, 5, 6, 9, 10, 11, 12, 13, 14, 18, 19, 27, 28]

1. Let $\mathbf{u} = (2, 3, -1)$, $\mathbf{w} = (1, -1, 1)$, and $3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = (-1, 2, 3)$. Find (a) $\mathbf{v}$ (b) $-2\mathbf{u} + 3\mathbf{v} - 5\mathbf{w}$.

   Solution:

   (a) Given

$$\begin{aligned} 3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = (-1, 2, 3) \Rightarrow \mathbf{v} =& \tfrac{1}{2}[3\mathbf{u} + 4\mathbf{w} - (-1, 2, 3)] \\ =& \tfrac{1}{2}[3(2, 3, -1) + 4(1, -1, 1) - (-1, 2, 3)] \\ =& \tfrac{1}{2}(11, 3, -2) \end{aligned}$$

   (b) $-2\mathbf{u} + 4\mathbf{v} - 5\mathbf{w} = (-4, -6, 2) + (22, 6, -4) - (5, -5, 5) = (13, 5, -7)$

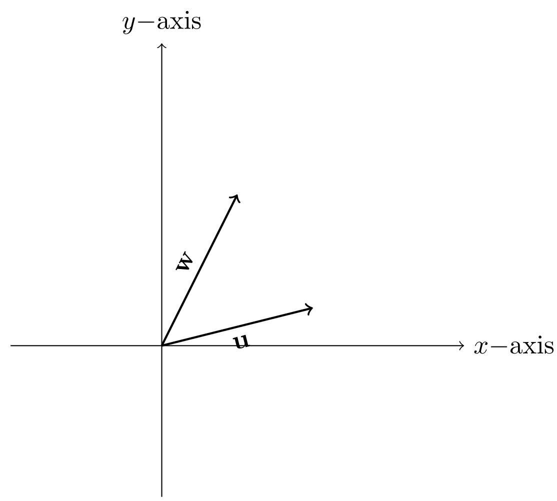
2. Given vectors **v** and **w** in diagram below, shade in all linear combinations $c\mathbf{u} + d\mathbf{w}$ for

(a) $0 \leq c \leq 1$ and $-1 \leq d \leq 0$

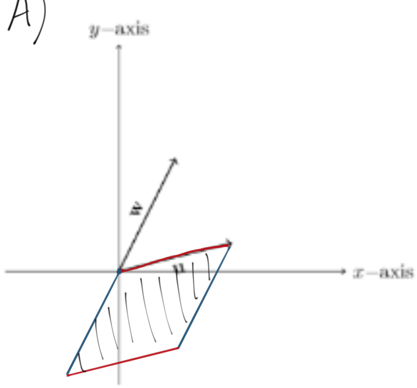
(b) $0 \leq c \leq 1$ and $1 \leq d$

(c) $0 \leq d \leq 1$ and $c \geq 1$.

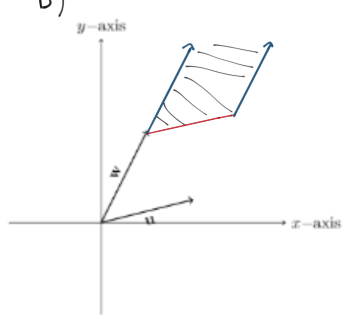
(You can use different shading styles in same picture for all three parts or can graph them separately in three different pictures)



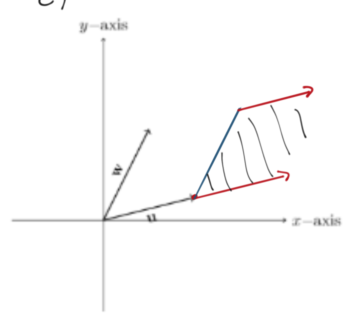
A)



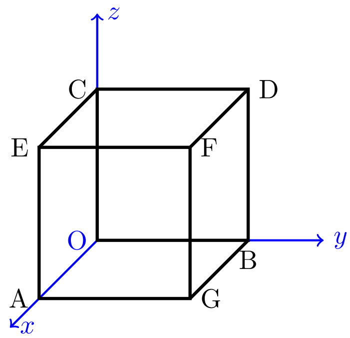
B)



c)



3. Let $\mathbf{0} = (0,0,0)$, $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ be vectors in $\mathbb{R}^3$. Let P, Q, R, S, T, U be the center of faces $OAGB, OBDC, OAEC, GBDF, FECD, AGFE$ respectively of the unit cube in the figure below. Write down the following vectors as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.



(a) $\overrightarrow{OP} =$

Solution: $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}((1,0,0) + (0,1,0)) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

(b) $\overrightarrow{OQ} =$

Solution: $\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OD} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{2}((0,1,0) + (0,0,1)) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$.

(c) $\overrightarrow{OR} =$

Solution: $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) = \frac{1}{2}((1,0,0) + (0,0,1)) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$.

(d) $\overrightarrow{OS} =$

Solution: $\overrightarrow{OS} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BF} = \overrightarrow{OB} + \frac{1}{2}(\overrightarrow{BG} + \overrightarrow{BD}) = (0,1,0) + \frac{1}{2}((1,0,0) + (0,0,1)) = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$.

(e) $\overrightarrow{OT} =$

Solution: $\overrightarrow{OT} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CF} = \overrightarrow{OC} + \frac{1}{2}(\overrightarrow{CD} + \overrightarrow{CE}) = (0,0,1) + \frac{1}{2}((0,1,0) + (1,0,0)) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}$.

(f) $\overrightarrow{OU} =$

Solution: $\overrightarrow{OU} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AF} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AE} + \overrightarrow{AG}) = (1,0,0) + \frac{1}{2}((0,0,1) + (0,1,0)) = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$.

4. Consider the unit cube as in the previous question. The face $OAGB$ can be described as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as

$$OAGB = \{c\mathbf{i} + d\mathbf{j} : 0 \leq c \leq 1, 0 \leq d \leq 1\}.$$

Geometrically describe the remaining five faces of the unit cube as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

(a) $OBDC =$

Solution: $OBDC = \{c\mathbf{j} + d\mathbf{k} : 0 \leq c \leq 1, 0 \leq d \leq 1\}$.

(b) $OAEC =$

Solution: $OAEC = \{c\mathbf{i} + d\mathbf{k} : 0 \leq c \leq 1, 0 \leq d \leq 1\}$.

(c) $GBDF =$

Solution: $GBDF = \{c\mathbf{i} + \mathbf{j} + d\mathbf{k} : 0 \leq c \leq 1, 0 \leq d \leq 1\}$.

(d) $FECD =$

Solution: $FECD = \{c\mathbf{i} + d\mathbf{j} + \mathbf{k} : 0 \leq c \leq 1, 0 \leq d \leq 1\}$.

(e) $AGFE =$

Solution: $AGFE = \{\mathbf{i} + c\mathbf{j} + d\mathbf{k} : 0 \leq c \leq 1, 0 \leq d \leq 1\}$.

5. Given three vectors $\mathbf{u} = (1, 1), \mathbf{v} = (1, -1)$, and $\mathbf{b} = (2, 4)$ in $\mathbb{R}^2$. Suppose $\mathbf{b}$ can be written as linear combination of $\mathbf{u}$ and $\mathbf{v}$ as

$$c\mathbf{u} + d\mathbf{v} = \mathbf{b}$$

(a) Write two equations in c and d corresponding to the vector equation $c\mathbf{u} + d\mathbf{v} = \mathbf{b}$.

Solution:

$$c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix},$$

$$\begin{cases} c + d = 2 \\ c - d = 4 \end{cases}$$

(b) Solve the equations in part (a) for c and d.

Solution:

$$\begin{cases} c = 3 \\ d = -1 \end{cases}$$

(c) Express $\mathbf{b}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$\mathbf{b} = 3\mathbf{u} - \mathbf{v}, \quad \textit{or} \quad \mathbf{b} = 3\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}$$

6. Let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (1, -1)$, be 2 given vectors in $\mathbb{R}^2$. Let $\mathbf{b} = (p, q)$ be any vector in $\mathbb{R}^2$.

(a) Write two equations in c and d corresponding to the vector equation $c\mathbf{u} + d\mathbf{v} = \mathbf{b}$.

Solution:

$$\begin{aligned} c\mathbf{u} + d\mathbf{v} &= \mathbf{b} \\ \implies c(1,1) + d(1,-1) &= (p,q) \\ \implies (c+d, c-d) &= (p,q) \end{aligned}$$

$\implies c + d = p$ and $c - d = q$

The 2 equations are $\begin{cases} c + d = p & (i) \\ c - d = q & (ii) \end{cases}$

(b) Solve the equations in part (a) for c and d.

Solution: Adding (i) and (ii), $2c = p + q \Rightarrow c = \frac{p+q}{2}$
Subtracting (ii) from (i), $2d = p - q \Rightarrow d = \frac{p-q}{2}$
Therefore $c = \frac{p+q}{2}$ and $d = \frac{p-q}{2}$

(c) Express $\mathbf{b}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ (if possible).

Solution: $\mathbf{b} = \left(\frac{p+q}{2}\right)\mathbf{u} + \left(\frac{p-q}{2}\right)\mathbf{v}$

7. Let $\mathbf{u}+\mathbf{v}=(3,4)$ and $\mathbf{u}-\mathbf{v}=(1,-2)$.

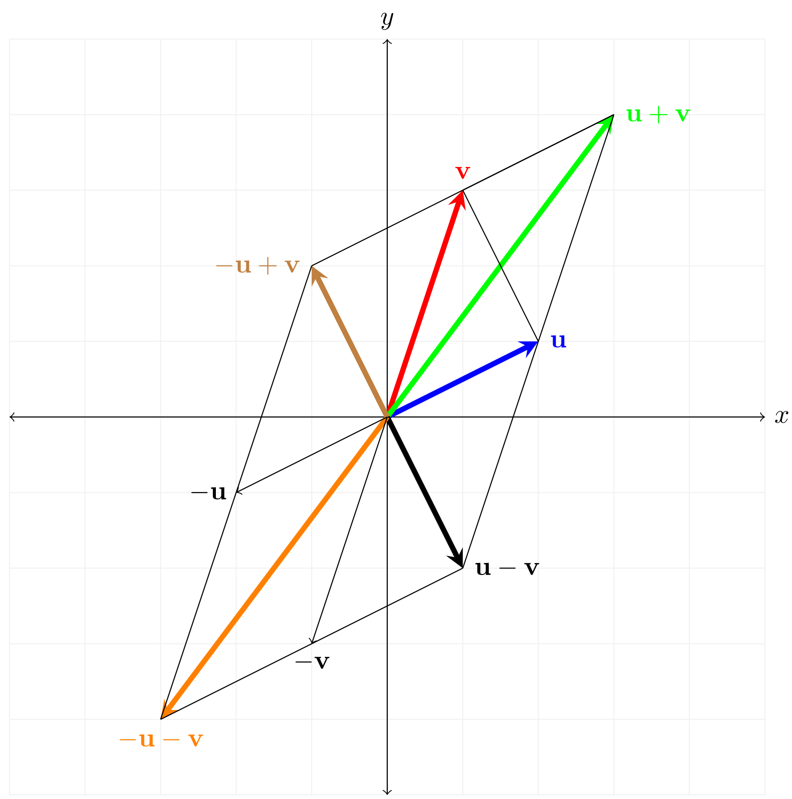
(a) Find $\mathbf{u}$ and $\mathbf{v}$

Solution

$$\begin{aligned}\mathbf{u} &= \tfrac{1}{2}[(\mathbf{u}+\mathbf{v})+(\mathbf{u}-\mathbf{v})]\\ &= \tfrac{1}{2}[(3,4)+(1,-2)] = \tfrac{1}{2}(4,2)\\ &= (2,1) \quad \text{and}\\ \mathbf{v} &= \tfrac{1}{2}[(\mathbf{u}+\mathbf{v})-(\mathbf{u}-\mathbf{v})]\\ &= \tfrac{1}{2}[(3,4)-(1,-2)] = \tfrac{1}{2}(2,6)\\ &= (1,3)\end{aligned}$$

(b) Use the vectors $\mathbf{u},\ \mathbf{v}, -\mathbf{u}, -\mathbf{v},\ \mathbf{u}+\mathbf{v},\ (\mathbf{u}-\mathbf{v}),\ (-\mathbf{u}+\mathbf{v}),\ (-\mathbf{u}-\mathbf{v})$ to label the following figure.

Solution



8. Consider a clock of radius 2 unit.

(a) Write down the 12 vectors in the component form that go from the center of the clock to the hours

$$1:00, 2:00, \cdots\cdots\cdots\cdots, 12:00.$$

Solution:
Let $r = 2$ and :
$\mathbf{v}_1$ denote to $1:00$ with angle $\theta = \frac{\pi}{3}$
$\mathbf{v}_2$ denote to $2:00$ with angle $\theta = \frac{\pi}{6}$
$\mathbf{v}_3$ denote to $3:00$ with angle $\theta = 0$
$\mathbf{v}_4$ denote to $4:00$ with angle $\theta = \frac{11\pi}{6}$
$\mathbf{v}_5$ denote to $5:00$ with angle $\theta = \frac{10\pi}{6}$
$\mathbf{v}_6$ denote to $6:00$ with angle $\theta = \frac{3\pi}{2}$
$\mathbf{v}_7$ denote to $7:00$ with angle $\theta = \frac{8\pi}{6}$
$\mathbf{v}_8$ denote to $8:00$ with angle $\theta = \frac{7\pi}{6}$
$\mathbf{v}_9$ denote to $9:00$ with angle $\theta = \pi$
$\mathbf{v}_{10}$ denote to $10:00$ with angle $\theta = \frac{5\pi}{6}$
$\mathbf{v}_{11}$ denote to $11:00$ with angle $\theta = \frac{2\pi}{3}$
$\mathbf{v}_{12}$ denote to $12:00$ with angle $\theta = \frac{\pi}{2}$
Then we can write those vectors in polar coordinates form $\mathbf{v} = (r\cos\theta, r\sin\theta)$ as follows:

$$\mathbf{v}_1 = (2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}) = (1, \sqrt{3}) \qquad \mathbf{v}_2 = (2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}) = (\sqrt{3}, 1)$$

$$\mathbf{v}_3 = (2\cos 0, 2\sin 0) = (2, 0) \qquad \mathbf{v}_4 = (2\cos\frac{11\pi}{6}, 2\sin\frac{11\pi}{6}) = (\sqrt{3}, -1)$$

$$\mathbf{v}_5 = (2\cos\frac{10\pi}{6}, 2\sin\frac{10\pi}{6}) = (1, -\sqrt{3}) \qquad \mathbf{v}_6 = (2\cos\frac{0\pi}{6}, 2\sin\frac{9\pi}{6}) = (0, -2)$$

$$\mathbf{v}_7 = (2\cos\frac{8\pi}{6}, 2\sin\frac{8\pi}{6}) = (-1, -\sqrt{3}) \qquad \mathbf{v}_8 = (2\cos\frac{7\pi}{6}, 2\sin\frac{7\pi}{6}) = (-\sqrt{3}, -1)$$

$$\mathbf{v}_9 = (2\cos\pi, 2\sin\pi) = (-2, 0) \qquad \mathbf{v}_{10} = (2\cos\frac{5\pi}{6}, 2\sin\frac{5\pi}{6}) = (-\sqrt{3}, 1)$$

$$\mathbf{v}_{11} = (2\cos\frac{4\pi}{6}, 2\sin\frac{4\pi}{6}) = (-1, \sqrt{3}) \qquad \mathbf{v}_{12} = (2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = (0, 2)$$

(b) What is the sum of those 12 vectors? Explain.
Solution:
The sum of those 12 vectors is zero vector, because 6 of them are the opposite of the others, for example $\mathbf{v}_1$ and $\mathbf{v}_7$ they have the same length but in the opposite direction so $\mathbf{v}_1 + \mathbf{v}_7 = 0$, and so on. Also it's clear if we add the components of the 12 vectors will will be the **zero vector** $\mathbf{0} = (0, 0)$.

9. Consider the same clock as in previous question.

   (a) Write down the 12 vectors in component form that go from 3 : 00 on the right to the hours

$$1:00, 2:00, \cdots\cdots\cdots, 12:00.$$

   Solution: If the vectors start form 3 : 00 on the right instead of the center of the clock, the first components will be less by 2 and the second components will remain same (because shifting origin to the 2 unit right does not change the $y-$coordinate but $x-$coordinate will be less by 2. So from Q.N.#8, the 12 vectors will be
   follows:

$$\begin{aligned} \mathbf{v}_1 &= (-1, \sqrt{3}) & \mathbf{v}_2 &= (\sqrt{3}-2, 1) \\ \mathbf{v}_3 &= (0, 0) & \mathbf{v}_4 &= (\sqrt{3}-2, -1) \\ \mathbf{v}_5 &= (-1, -\sqrt{3}) & \mathbf{v}_6 &= (-2, -2) \\ \mathbf{v}_7 &= (-3, -\sqrt{3}) & \mathbf{v}_8 &= (-\sqrt{3}-2, -1) \\ \mathbf{v}_9 &= (-4, 0) & \mathbf{v}_{10} &= (-\sqrt{3}-2, 1) \\ \mathbf{v}_{11} &= (-3, \sqrt{3}) & \mathbf{v}_{12} &= (-2, 2) \end{aligned}$$

   (b) What is the sum of the 12 vectors? Explain

   Solution: Since the second components have not changed from Q.N.#8 and the first component have become less by 2 for all 12 vectors, the sum has to be $(-24, 0)$.

10. For each set of vectors in $\mathbb{R}^3$ given below. Describe geometrically the set of all linear combinations (a line or plane or all of $\mathbb{R}^3$).

(a) $\{(0,1,-3),(0,-2,6),(4,2,-6)\}$

Solution: Denote $\mathbf{u}=(0,1,-3)$, $\mathbf{v}=(0,-2,6)$, $\mathbf{w}=(4,2,-6)$. Note that $\mathbf{v}=-2\mathbf{u}$ and $\mathbf{u},\mathbf{w}$ are linearly independent (not parallel to each other), thus the set of all linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$ is two-dimensional, *i.e.* a plane.

(b) $\{(2,1,0),(1,1,1),(4,3,2)\}$

Solution: Denote $\mathbf{u}=(2,1,0)$, $\mathbf{v}=(1,1,1)$, $\mathbf{w}=(4,3,2)$. It is easy to see that $\mathbf{w}=\mathbf{u}+2\mathbf{v}$ and $\mathbf{u},\mathbf{v}$ are linearly independent (not parallel to each other), hence the set of all linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$ is two-dimensional, *i.e.* a plane.

(c) $\{(2,-3,1),(-4,6,-2),(-10,15,-5)\}$

Solution: Denote $\mathbf{u}=(2,-3,1)$, $\mathbf{v}=(-4,6,-2)$, $\mathbf{w}=(-10,15,-5)$. Note that $\mathbf{v}=-2\mathbf{u}$ and $\mathbf{w}=-5\mathbf{u}$, thus the set of all linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$ is one-dimensional, *i.e.* a line.