MATH 2418: Linear Algebra

Assignment 12 (Sections 6.2 and 6.4)

Due: `May 1st, 2019` Term: `Spring 2019`

[First Name] [Last Name] [Net ID]

Recommended Text Book Problems (do not turn in): [Section 6.2: 1, 2, 7, 9, 15, 21, 24, 29. Section 6.4: 1, 5, 7, 9, 14, 25, 28, 34.] Solutions to these problems are available at *math.mit.edu/linearalgebra*

1. Factor the following matrix into $A = X\Lambda X^{-1}$.

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 7 \end{bmatrix}$$

(a) [2 points] Find the eigenvalues of A.

Solution:
A is triangular and hence eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 7$, the diagonal elements.

(b) [4 points] Find an eigenvector associated with each eigenvalue of A.

Solution:
Let $\mathbf{x}$ be an eigenvector corresponding to λ. Solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$

$$\lambda = 3 : (A - 3I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = -2y,\ y = 1, x = -2 \Rightarrow \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = 7 : (A - 3I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -4 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = 0,\ y = 1 \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) [4 points] Construct matrices X and Λ so that $A = X\Lambda X^{-1}$ and Λ is diagonal.

Solution:

$$X = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

2. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) [2 points] Show that the only eigenvalue of A is $\lambda = 2$.

(b) [2 points] What is the rank of the matrix $A - 2I$?

(c) [2 points] What is the dimension of the nullspace of $A - 2I$?

(d) [2 points] How many linearly independent eigenvectors can be found for the matrix A?

(e) [2 points] Can the matrix A be factored into the form $A = X\Lambda X^{-1}$? (yes or no)

Solution:

(a) Will find the roots of the polynomial $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 1-\lambda \\ 0 & -1 \end{vmatrix}$$
$$= (3-\lambda)[(1-\lambda)(2-\lambda)+1] - 1 = (3-\lambda)(1-\lambda)(2-\lambda) + (3-\lambda) - 1$$
$$= (2-\lambda)[(3-\lambda)(1-\lambda)+1] = (2-\lambda)(\lambda-2)^2 = (2-\lambda)^3 = 0$$

Then $\lambda = 2$ is the only eigenvalue of A.

(b) We put $A - 2I$ into upper triangle form:

$$A - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the rank of the matrix $A - 2I$ equals 2 (2 pivots).

(c) The dimension of the nullspace is $n - r = 3 - 2 = 1$

(d) Since we have only one distinct eigenvalue and the corresponding nullspace has dimension one, then we have only one linearly independent eigenvector.

(e) No. It can not be factorized, because there are no 3 linearly independent eigenvectors.

3. Let M be the Markov matrix $M = \begin{bmatrix} .2 & 0 \\ .8 & 1 \end{bmatrix}$.

   (a) [4 points] Factor M in the form $M = X\Lambda X^{-1}$.

   (b) [4 points] Show that $M^k = \begin{bmatrix} (.2)^k & 0 \\ 1-(.2)^k & 1 \end{bmatrix}$.

   (c) [2 points] Calculate $\lim_{k\to\infty} M^k$.

Solution:

a) Observe that the eigenvalues of M are 0.2 and 1 (the eigenvalues of a triangular matrix are its diagonal entries). Then we compute the eigenvectors corresponding to the eigenvalues: $M - 0.2I = \begin{bmatrix} 0 & 0 \\ 0.8 & 0.8 \end{bmatrix}$, which has null space spanned by $(1, -1)$. On the other hand, $M - 1I = \begin{bmatrix} -0.8 & 0 \\ 0.8 & 0 \end{bmatrix}$, which has null space spanned by $(0, 1)$. Choosing to have $\lambda = 0.2$ as our first eigenvalue and $\lambda = 1$ as our second (this choice is arbitrary), we then have $\Lambda = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ (eigenvectors go to the columns of X corresponding to their eigenvalue, the order of which we arbitrarily chose).

b) $M^k = (X\Lambda X^{-1})^k = X\Lambda X^{-1}X\Lambda X^{-1}\cdots X\Lambda X^{-1}$ (k times). Then we cancel the inner $X^{-1}X$ terms to give $\cdots = X\Lambda^k X^{-1} = \begin{bmatrix} (.2)^k & 0 \\ 1-(.2)^k & 1 \end{bmatrix}$.

c) Taking limits of each term in the matrix from part b), we obtain $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Notice that the eigenvalues of this matrix are the limits of λ^k as k goes to infinity, where the λ are the eigenvalues of M. The eigenvectors are identical to the eigenvectors of M.

4. [10 points] Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Write S in the form $S = Q\Lambda Q^T$.

Solution:

Matrix Λ is a diagonal matrix of the eigenvalues of S, and the columns of Q are unit eigenvectors of S.

First we need to find the eigenvalues of S, i.e., solve $\det(S - \lambda I) = 0$.

$$\det \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = (\lambda+1)^2 - 1 = \lambda^2 + 2\lambda = \lambda(\lambda+2) = 0.$$

Thus, eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -2$.

Now, we need to find unit eigenvectors.

(a) $\lambda_1 = 0$:

$$(S - 0I)q_1 = 0 \quad \Rightarrow \quad Sq_1 = 0 \quad \Rightarrow \quad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) $\lambda_2 = -2$:

$$(S + 2I)q_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} q_2 = 0 \quad \Rightarrow \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore, $\Lambda = \text{diag}(0, -2)$ and $Q = [q_1 | q_2]$,

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

5. [10 points] Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Write S as the sum of two rank one matrices using the spectral form $S = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T$.

Solution:

First let us find the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - (-1)^2 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$$

So for $\lambda = 4$, we have:

$$A - 4I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\Rightarrow \ker(A - 4I) = \text{span}\{(1, -1)\}$$

For $l\lambda = 2$, we have:

$$A - 2I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \ker(A - 2I) = \text{span}\{(1, 1)\}$$

So we can normalize these vectors to get our Q matrix.

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\Rightarrow Q^{-1} = Q^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

So

$$S = Q\Lambda Q^T$$

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Note that we can also write S in terms of the outer products and eigenvector scaling, i.e.

$$S = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T$$

where $\mathbf{q}_i$ is the normalized eigenvector associated with λ_i. So

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = 4 * \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} + 2 * \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

6. The matrix $S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalues 1, d, and $-d$.

(a) [3 points] Find a vector $\mathbf{q}_1$ that satisfies $S\mathbf{q}_1 = \mathbf{q}_1$ and $\mathbf{q}_1^T\mathbf{q}_1 = 1$.

Solution:
By observing the last column of S, it is readily seem that $\mathbf{q}_1^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, hence it is verified that $S\mathbf{q}_1 = \mathbf{q}_1$ and $\mathbf{q}_1^T\mathbf{q}_1 = 1$

(b) [3 points] Find a vector $\mathbf{q}_2$ that satisfies $S\mathbf{q}_2 = d\mathbf{q}_2$ and $\mathbf{q}_2^T\mathbf{q}_2 = 1$.

Solution:
Calculating special solutions for $(S - dI)\mathbf{x} = \mathbf{0}$ and considering $d \notin \{0, 1\}$, we have

$$\begin{bmatrix} -d & d & 0 \\ d & -d & 0 \\ 0 & 0 & 1-d \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence we take $\mathbf{q}_2 = \dfrac{\mathbf{x}}{\|\mathbf{x}\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$

(c) [3 points] Find a vector $\mathbf{q}_3$ that satisfies $S\mathbf{q}_3 = -d\mathbf{q}_3$ and $\mathbf{q}_3^T\mathbf{q}_3 = 1$.

Solution:
Calculating special solutions for $(S + dI)\mathbf{x} = \mathbf{0}$ and considering $d \notin \{0, 1\}$, we have

$$\begin{bmatrix} d & d & 0 \\ d & d & 0 \\ 0 & 0 & 1+d \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Hence we take $\mathbf{q}_3 = \dfrac{\mathbf{x}}{\|\mathbf{x}\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$

(d) [1 point] Show that the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are orthogonal to one another.

Solution:
$\mathbf{q}_1^T\mathbf{q}_2 = 0.\left(\frac{1}{\sqrt{2}}\right) + 0.\left(\frac{1}{\sqrt{2}}\right) + 1.(0) = 0$, $\mathbf{q}_1^T\mathbf{q}_3 = 0.\left(-\frac{1}{\sqrt{2}}\right) + 0.\left(\frac{1}{\sqrt{2}}\right) + 1.(0) = 0$,
$\mathbf{q}_2^T\mathbf{q}_3 = \left(\frac{1}{\sqrt{2}}\right).\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{\sqrt{2}}\right) + 0.(0) = -\frac{1}{2} + \frac{1}{2} = 0$