MATH 2418: Linear Algebra

Assignment 12 (Sections 6.2 and 6.4)

Due: `May 1st, 2019` Term: `Spring 2019`

[First Name] [Last Name] [Net ID]

Recommended Text Book Problems (do not turn in): [Section 6.2: 1, 2, 7, 9, 15, 21, 24, 29. Section 6.4: 1, 5, 7, 9, 14, 25, 28, 34.] Solutions to these problems are available at *math.mit.edu/linearalgebra*

1. Factor the following matrix into $A = X\Lambda X^{-1}$.

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 7 \end{bmatrix}$$

   (a) [2 points] Find the eigenvalues of A.

   (b) [4 points] Find an eigenvector associated with each eigenvalue of A.

   (c) [4 points] Construct matrices X and Λ so that $A = X\Lambda X^{-1}$ and Λ is diagonal.

2. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

   (a) [2 points] Show that the only eigenvalue of A is $\lambda = 2$.

   (b) [2 points] What is the rank of the matrix $A - 2I$?

   (c) [2 points] What is the dimension of the nullspace of $A - 2I$?

   (d) [2 points] How many linearly independent eigenvectors can be found for the matrix A?

   (e) [2 points] Can the matrix A be factored into the form $A = X\Lambda X^{-1}$? (yes or no)

3. Let M be the Markov matrix $M = \begin{bmatrix} .2 & 0 \\ .8 & 1 \end{bmatrix}$.

(a) [4 points] Factor M in the form $M = X\Lambda X^{-1}$.

(b) [4 points] Show that $M^k = \begin{bmatrix} (.2)^k & 0 \\ 1-(.2)^k & 1 \end{bmatrix}$.

(c) [2 points] Calculate $\lim_{k\to\infty} M^k$.

4. [10 points] Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Write S in the form $S = Q\Lambda Q^T$.

5. [10 points] Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Write S as the sum of two rank one matrices using the spectral form $S = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T$.

6. The matrix $S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalues 1, d, and $-d$.

   (a) [3 points] Find a vector $\mathbf{q}_1$ that satisfies $S\mathbf{q}_1 = \mathbf{q}_1$ and $\mathbf{q}_1^T\mathbf{q}_1 = 1$.

   (b) [3 points] Find a vector $\mathbf{q}_2$ that satisfies $S\mathbf{q}_2 = d\mathbf{q}_2$ and $\mathbf{q}_2^T\mathbf{q}_2 = 1$.

   (c) [3 points] Find a vector $\mathbf{q}_3$ that satisfies $S\mathbf{q}_3 = -d\mathbf{q}_3$ and $\mathbf{q}_3^T\mathbf{q}_3 = 1$.

   (d) [1 point] Show that the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are orthogonal to one another.