MATH 2418: Linear Algebra

Assignment 11 (Sections 5.3 and 6.1)

Due: April 24th, 2019 Term: Spring 2019

[First Name] [Last Name] [Net ID]

Recommended Text Book Problems (do not turn in): [Section 5.3: 1, 5, 6, 7, 14, 18, 31, 32. Section 6.1: 2, 3, 4, 9, 15, 16, 17, 24.] Solutions to these problems are available at *math.mit.edu/linearalgebra*

1. Solve the following linear system for $\mathbf{x} = (x_1, x_2, x_3)$ using Cramer's Rule.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) [1 point] $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -3 \end{vmatrix} =$

Solution:

$$\det A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -3 \end{vmatrix} = 1.(-1)^{1+1} \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} = -5$$

(b) [3 points] $x_1 =$

Solution:

$\det B_1 = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 3 & 2 & -3 \end{vmatrix} = 0$, notice the last column is the multiple of the first one. Applying the Cramer's Rule, we have $x_1 = \dfrac{\det B_1}{\det A} = 0$

(c) [3 points] $x_2 =$

Solution:

$$\det B_2 = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{vmatrix} = 0 \Rightarrow x_2 = \frac{\det B_2}{\det A} = 0$$

(d) [3 points] $x_3 =$

Solution:

$$\det B_3 = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{vmatrix} = -\det A = 5 \Rightarrow x_3 = \frac{\det B_3}{\det A} = \frac{5}{-5} = -1$$

2. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 2 & -1 & 1 \end{bmatrix}$, find the determinant, cofactor matrix, and inverse.

(a) [2 points] Find $\det A$.

Solution:

$$\det(A) = (-1)^{(1+1)} * 1 * \det \begin{bmatrix} 4 & 5 \\ -1 & 1 \end{bmatrix} + (-1)^{(2+1)} * 0 * \det \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + (-1)^{(3+1)} * 2 * \det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\det(A) = 1 * 1 * 9 + (-1) * 0 * 5 + 1 * 2 * (-2) = 9 - 4 = 5$$

(b) [4 points] Find the cofactor matrix C.

Solution:

Using the formula:

$$C_{ij} = (-1)^{(i+j)} * \det(M_{ij})$$

Where M_{ij} is the matrix where the ith row and jth colum are removed from matrix A, we get that:

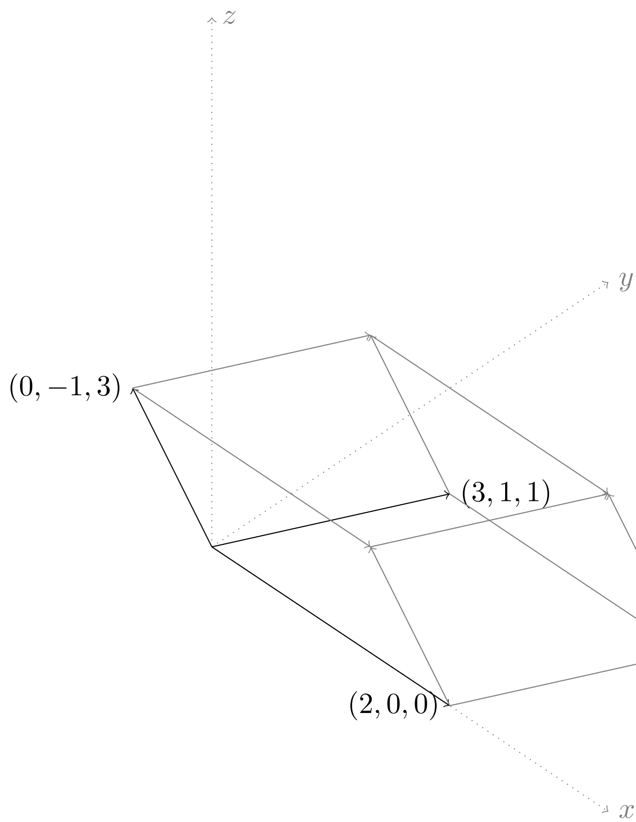
$$C = \begin{bmatrix} 9 & 10 & -8 \\ -5 & -5 & 5 \\ -2 & -5 & 4 \end{bmatrix}$$

(c) [4 points] Use the cofactor matrix to find A^{-1}.

Solution:

$$A^{-1} = C^T / \det(A) = 1/5 * \begin{bmatrix} 9 & -5 & -2 \\ 10 & -5 & -5 \\ -8 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 9/5 & -1 & -2/5 \\ 2 & -1 & -1 \\ -8/5 & 1 & 4/5 \end{bmatrix}$$

3. [10 points] Find the volume of a box whose edges are the vectors $\mathbf{a}_1 = (2,0,0)$, $\mathbf{a}_2 = (3,1,1)$, and $\mathbf{a}_3 = (0,-1,3)$.



Solution:

The volume is given by the triple product $(\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3$ which is:

$$\begin{vmatrix} 0 & -1 & 3 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{vmatrix} = (-1)^{2+1} \cdot 2 \cdot \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = 8$$

4. Based on the information given below, identify an eigenvalue of the matrix A.

(a) [2 points] $A = \begin{bmatrix} .1 & .9 \\ .5 & .5 \end{bmatrix}$ and $\begin{bmatrix} .1 & .9 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is an eigenvalue of A?

Solution:
Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the above equation gives us $A\mathbf{x} = 1\mathbf{x}$, so 1 is an eigenvalue of A.

(b) [2 points] $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. What is an eigenvalue of A?

Solution:
Let $x = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}$, then the above equation gives us $A\mathbf{x} = 0\mathbf{x}$, so 0 is an eigenvalue of A.

(c) [2 points] $A = \begin{bmatrix} 7 & 9 & 5 \\ 8 & 3 & 5 \\ -8 & 1 & -6 \end{bmatrix}$ and $\begin{bmatrix} 7 & 9 & 5 \\ 8 & 3 & 5 \\ -8 & 1 & -6 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -8 \end{bmatrix}$. What is an eigenvalue of A?

Solution:
Similar to (a), 1 is an eigenvalue of A

(d) [2 points] $A = \begin{bmatrix} 18 & -3 & 4 \\ -3 & 8 & 0 \\ 8 & 2 & 17 \end{bmatrix}$ and $\begin{bmatrix} 18 & -3 & 4 \\ -3 & 8 & 0 \\ 8 & 2 & 17 \end{bmatrix} \begin{bmatrix} -10 \\ 6 \\ 17 \end{bmatrix} = \begin{bmatrix} -130 \\ 78 \\ 221 \end{bmatrix}$. What is an eigenvalue of A?

Solution:
Note that $\begin{bmatrix} -130 \\ 78 \\ 221 \end{bmatrix} = 13 \cdot \begin{bmatrix} -10 \\ 6 \\ 17 \end{bmatrix}$, so the above equation gives us $A\mathbf{x} = 13\mathbf{x}$, so 13 is an eigenvalue of A.

(e) [2 points] $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 3 \\ -2\sqrt{3} \\ 3 \\ -\sqrt{3} \end{bmatrix}$. What is an eigenvalue of A?

Solution:
Note that $\begin{bmatrix} -\sqrt{3} \\ 3 \\ -2\sqrt{3} \\ 3 \\ -\sqrt{3} \end{bmatrix} = -\sqrt{3} \cdot \begin{bmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{bmatrix}$, so the above equation gives us $A\mathbf{x} = -\sqrt{3}\mathbf{x}$, so $-\sqrt{3}$

is an eigenvalue of A.

5. Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$.

(a) [2 points] Solve the equation $\det(A - \lambda I) = 0$ for its three roots, λ_1, λ_2, and λ_3.

Solution:
$det \begin{bmatrix} -\lambda & -1 & 0 \\ -1 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{bmatrix} = 0$, computing the determinant we got an equation
$-\lambda(\lambda^2 - 2) = 0$ with roots: $\lambda_1 = 0$, $\lambda_2 = \sqrt{2}$, $\lambda_3 = -\sqrt{2}$

(b) [2 points] Solve the equation $(A - \lambda_1 I)\mathbf{x} = 0$ for an eigenvector associated with the eigenvalue λ_1.

Solution:
for $\lambda_1 = 0$:
$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, solving this system, the first eigenvector is $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(c) [2 points] Solve the equation $(A - \lambda_2 I)\mathbf{x} = 0$ for an eigenvector associated with the eigenvalue λ_2.

Solution:
for $\lambda_2 = \sqrt{2}$:
$\begin{bmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, solving this system, the second eigenvector is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

(d) [2 points] Solve the equation $(A - \lambda_3 I)\mathbf{x} = 0$ for an eigenvector associated with the eigenvalue λ_3.

Solution:
for $\lambda_3 = -\sqrt{2}$:
$\begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, solving this system, the third eigenvector is $\mathbf{x}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

(e) [2 points] For each eigenvector found in parts (b), (c) and (d), calculate the product $A\mathbf{x}$ and verify that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$.

Solution:

For each eigenvector we need to show that $A\mathbf{x} = \lambda\mathbf{x}$.
For vector from part b):
$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Indeed, multiple with $\lambda_1 = 0$.

For vector from part c):

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ -2 \\ -\sqrt{2} \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$ Indeed, multiple with $\lambda_2 = -\sqrt{2}$.

For vector from part d):

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ -2 \\ \sqrt{2} \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$ Indeed, multiple with $\lambda_3 = \sqrt{2}$.

6. Find the eigenvalues of the following matrices and answer the questions below.

   (a) [2 points] $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

   (b) [2 points] $A^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

   (c) [2 points] $A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 9 \end{bmatrix}$

   (d) [1 point] What is the sum of the eigenvalues of A?

   (e) [1 point] What is the product of the eigenvalues of A?

   (f) [1 point] What is the sum of the eigenvalues of $A^T A$?

   (g) [1 point] What is the product of the eigenvalues of $A^T A$?

Solution:

(a).

The characteristic equation is:

$$|A - \lambda\ I| = 0 \quad \Longrightarrow \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 0 \implies (2-\lambda)\,(3-\lambda) - 0 = 0 \implies \lambda = 2,\ 3$$

Therefore, eigenvalues of A are $\lambda = 2,\ 3$

(b).

The characteristic equation is:

$$\left|A^T - \lambda\ I\right| = 0 \quad \Longrightarrow \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0 \implies (2-\lambda)\,(3-\lambda) - 0 = 0 \implies \lambda = 2,\ 3$$

Therefore, eigenvalues of A^T are $\lambda = 2,\ 3$

(c).

The characteristic equation is:

$$\left|A^T A - \lambda\ I\right| = 0 \quad \Longrightarrow \begin{vmatrix} 5-\lambda & 3 \\ 3 & 9-\lambda \end{vmatrix} = 0 \implies (5-\lambda)\,(9-\lambda) - 9 = 0 \implies \lambda^2 - 14\lambda + 36 = 0$$

$$\implies \lambda = 7 + \sqrt{13},\ 7 - \sqrt{13}$$

Therefore, eigenvalues of $A^T A$ are $\lambda = 7 + \sqrt{13},\ 7 - \sqrt{13}$

(d).

The sum of the eigenvalues of A is : $2 + 3 = 5$

(e).

The product of the eigenvalues of A is : $2 \cdot 3 = 6$

(f).

The sum of the eigenvalues of $A^T A$ is : $7 + \sqrt{13} + 7 - \sqrt{13} = 14$

(g).

The product of the eigenvalues of $A^T A$ is : $(7 + \sqrt{13}) \cdot (7 - \sqrt{13}) = 36$