MATH 2418: Linear Algebra

Assignment 10 (sections 5.1, 5.2)

Due: `April 17, 2019` Term: `Spring, 2019`

[First Name] [Last Name] [Net ID]

Suggested problems (do not turn in):Section 5.1: 1, 2, 3, 4, 5, 7, 8, 9, 12, 13, 15, 16, 17, 18, 21, 22, 23, 28; Section 5.2: 1, 2, 4, 5, 6, 7, 9, 12, 13, 16, 19, 20, 23, 24. Note that solutions to these suggested problems are available at *math.mit.edu/linearalgebra*

1. [10 points] Compute the determinants of the matrices A, B and AB.

$$A = \begin{bmatrix} 0 & 0 & -3 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 11 & 9 & 7 \\ 0 & 13 & 2 \\ 0 & 7 & 1 \end{bmatrix}$$

Solution:

For $\det A$ we use the cofactor expansion along row 2 we have

$$\begin{aligned} \det A &= \sum_{j=1}^{3} a_{2j} C_{2j} \\ &= -2(-1)^3 \begin{vmatrix} 0 & -3 \\ -1 & 0 \end{vmatrix} \\ &= -6 \end{aligned}$$

For $\det B$ we use the cofactor expansion along column 1.

$$\begin{aligned} \det B &= \sum_{i=1}^{3} a_{i1} C_{i1} \\ &= 11(-1)^2 \begin{vmatrix} 13 & 2 \\ 7 & 1 \end{vmatrix} \\ &= -11 \end{aligned}$$

For $\det(AB)$ we have,

$$\det(AB) = \det(A)\det(B) = -6(-11) = 66$$

2. [10 points] Let 3×3 matrices A and B have determinants -1 and 8 correspondingly.

(a) (3 points) Find determinant of $(4A^T)B^{-1}$.

(b) (3 points) Find determinant of B^2A^{-1}.

(c) (3 points) Find a scalar $c \in \mathbb{R}$ such that $\det[(cA)B] = 1$.

(d) (1 point) Find determinant of A^{-2017}.

Solution:

(a) $\det\left[(4A^T)B^{-1}\right] = \det\left(4A^T\right)\det B^{-1} = \dfrac{4^3 \det A^T}{\det B} = \dfrac{4^3 \det A}{\det B} = \dfrac{64 \times -1}{8} = -8$

(b) $\det\left[B^2A^{-1}\right] = \det B^2 \det A^{-1} = \dfrac{[\det B]^2}{\det A} = \dfrac{64}{-1} = -64$

(c) $1 = \det[(cA)B] = \det(cA)\det B = c^3 \det A \det B = -8c^3$, thus $c^3 = -\dfrac{1}{8}$. i.e $c = -\dfrac{1}{2}$

(d) $\det[A^{-2017}] = \dfrac{1}{\det A^{2017}} = \dfrac{1}{(-1)^{2017}} = -1$

3. [10 points] Compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 4 & 0 \\ -4 & 2 & 0 & 0 \end{bmatrix}$$

Solution:

Since the elimination steps do not change the determinant, we can add twice the first row to the second row and determinant will not change. Therefore,

$$\det B = \det \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ -4 & 2 & 0 & 0 \end{bmatrix}.$$

Next, we will compute the latter determinant by the co-factor expansion along the second row:

$$\det \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ -4 & 2 & 0 & 0 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 4 \\ -4 & 2 & 0 \end{bmatrix} = 1(0 - 2 \cdot 4) = -8.$$

4. [10 points] Let $A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}$. Find the determinant of the matrix $A^T B$.

Solution:

Note that both A and B are orthogonal matrices, so we have that their determinants must be either 1 or -1.

$$\det(A) = (-1)^{(3+3)} * \det \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = 1 * (2/4 - (-2/4)) = 1$$

$$\det(B) = (-1)^{(1+1)} * \det \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} = 1 * (3/4 - (-1/4)) = 1$$

So

$$\det(A^T B) = \det(A^T) * \det(B) = \det(A) * \det(B) = 1 * 1 = 1$$

5. [10 points] Find the determinant of the matrices P, P^2 and P^3.

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution:

Notice that:

$$P = P_{24}P_{45}P_{13}P_{34}P_{45}I_5$$

Then

$$\det P = \det(P_{24}P_{45}P_{13}P_{34}P_{45}) \det I_5 = (-1)^5 \times 1 = -1$$

And

$$\det(P^2) = (-1)^2 = 1, \quad \det(P^3) = (-1)^3 = -1$$

6. [10 points] True or False? Circle your answer and **provide a justification** for your choice.

(a) **T** **F:** If matrix A has one of its diagonal elements equal to zero, then $\det(A) = 0$.

Solution:
Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ which has one zero element in the diagonal. Note that $det(A) = -1 \neq 0$.

(b) **T** **F:** Absolute value of the determinant of the orthogonal matrix equals to 1.

Solution:
Let Q be an orthogonal matrix, then $Q^T Q = I$. So
$det(Q^T Q) = det(I) \implies det(Q^T)det(Q) = 1 \implies det(Q)det(Q) = 1 \implies det(Q)^2 = 1 \implies |det(Q)| = 1$.

(c) **T** **F:** If $\det B = \det B^{-1}$ then $B = I$.

Solution:
Let $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $B^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that $det(B) = det(B^{-1}) = 1$, but $B \neq I$.

(d) **T** **F:** If $\det(2C) = \det(3C)$ then C is not invertible.

Solution:
If C is an $n \times n$ matrix, then $det(2C) = det(3C) \implies 2^n det(C) = 3^n det(C)$
$\implies (2^n - 3^n)det(C) = 0$. Notice that $n \neq 0$, so $2^n - 3^n \neq 0$, hence $det(C) = 0$, it follows that C is not invertible.

(e) **T** **F:** There are no matrix D such that $\det(D^{-1}) = 0$.

Solution:
Suppose we have a matrix D such that $\det(D^{-1}) = 0$, then $det(D) = \frac{1}{det(D^{-1})} = \frac{1}{0}$ which is undefined, therefore, such matrix D does not exist.