① Recall: Let A be a $m \times n$ matrix

Course Evaluation in eLearning

• Singular Value Decomposition (SVD) of A

$$A = U \Sigma V^T.$$

$$= [u_1, \cdots, u_r, u_{r+1}, \cdots, u_m] \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, σ_i's are singular values of A.

r is the rank of A

1° $A^T A v_i = \sigma_i^2 v_i$, $i = 1, \cdots, r$.

v_i's are unit eigenvectors of $A^T A$

σ_i^2's are eigenvalues of $A^T A$

2° $Av_1 = \sigma_1 u_1$, $Av_2 = \sigma_2 u_2$, $\cdots$, $Av_r = \sigma_r u_r$ (1)

3° $u_{r+1}, \cdots, u_m$ is an orthonormal basis for $N(A^T)$.

$v_{r+1}, \cdots, v_n$ —————— $N(A)$.

Ex 1. Find a full SVD of $A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

$$S = A^T A = \begin{bmatrix} 5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{bmatrix}$$

$\det(S - \lambda I) = (\lambda - 10)\lambda^2 \Rightarrow \sigma_1^2 = 10$

Thus, $\sigma_1 = \sqrt{10}$

$(S - \sigma_1^2 I)x = 0 \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$Ax = 0 \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$A^T x = 0 \Rightarrow u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$U = [u_1, u_2] = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = [v_1, v_2, v_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Ex 2. Find a full SVD of $A = \begin{bmatrix} -1 & -1 \\ 1 & -2 \\ -1 & -1 \end{bmatrix}$

$$S = A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\det(S - \lambda I) = (\lambda - 6)(\lambda - 3) = 0 \Rightarrow \sigma_1^2 = 6, \quad \sigma_2^2 = 3$$

Thus, $\sigma_1 = \sqrt{6}$, $\sigma_2 = \sqrt{3}$

$$(S - \sigma_1^2 I)x = 0 \Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(S - \sigma_2^2 I)x = 0 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$Av_2 = \sigma_2 u_2 \Rightarrow u_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$r = 2$, $N(A) = \{0\}$

$$A^T u_3 = 0 \Rightarrow u_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$U = [u_1, u_2, u_3] = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$V = [v_1, v_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$