An Algorithm for Contraction of an Ore Ideal

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Krattenthaler's problem

Conjecture: Let $(a_n)_{\geq 0}$ and $(b_n)_{\geq 0}$ be two P-recursive sequences over the integers with leading coefficient n. Show that $(n! a_n b_n)_{\geq 0}$ is also a P-recursive sequence over the integers with leading coefficient n.

Example for Krattenthaler's problem

Consider the following P-recursive sequences:

$$\begin{aligned} na_n &= (31n-6)a_{n-1} + (49n-110)a_{n-2} + (9n-225)a_{n-3} \\ nb_n &= (4n+13)b_{n-1} + (69n-122)b_{n-2} + (36n-67)b_{n-3} \end{aligned}$$

The minimal recurrence for $c_n := n! a_n b_n$ is:

$$\alpha n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-9}$$

where $\alpha \in \mathbb{Z}[n]$, $\deg_n(\alpha) = 20$.

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Known algorithms find:

$$\beta n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-10}$$

where β is a 853-digit integer.

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Our algorithm finds:

$$1 n c_n = (\cdots)c_{n-1} + \ldots + (\cdots)c_{n-14}$$

Ore algebra (shift case)

Consider the recurrence equation:

$$f(n+1) - (n+1)f(n) = 0.$$

Using $\mathbb{Z}[n][\partial]$ with $\partial \bullet f(n) := f(n+1)$, $n \bullet f(n) := n \cdot f(n)$

$$[\partial - (n+1)] \bullet f = 0.$$

- L in $\mathbb{Z}[n][\partial]$ is called a recurrence operator of f if $L \bullet f = 0$.
- Assume $L = l_0 + \ldots + l_r \partial^r$, we call $\deg_\partial(L) := r$ the order of L, $\mathrm{lc}_\partial(L) := l_r$ the leading coefficient of L.
- T is called a left multiple of L if $T = PL$, where $P \in \mathbb{Q}(n)[\partial]$.

Motivation

Example 1 Consider the recurrence operator of $u(n)$:

$$L = (1+16n)^2\partial^2 - (224+512n)\partial - (1+n)(17+16n)^2$$

Question: Assume $u(0), u(1) \in \mathbb{Z}$, whether or not $u(n) \in \mathbb{Z}$, for each $n \in \mathbb{N}$?

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Our algorithm finds another left multiple of L:

$$\bar{T} := 1\partial^3 + \text{ lower terms } \in \mathbb{Z}[n][\partial]$$

Answer: Yes, $u(n)$ is an integer sequence.

Desingularization

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

- Assume $p \mid \operatorname{lc}_\partial(L)$. $T \in \mathbb{Z}[n][\partial]$ is a *p*-removed operator for L of order k if
  - T is a left multiple of L, $\deg_\partial(T) = k$.
  - $\operatorname{lc}_\partial(T) = ag(n)$, where $a \in \mathbb{Z}$, g is primitive, such that $g \mid \frac{1}{p} \operatorname{lc}_\partial(L)$.

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- If $\deg(g)$ is minimal (...), we call T is weakly desingularized operator (of order k).
- If $\deg(g)$ and a are minimal (...), we call T is strongly desingularized operator (of order k).

Desingularization

Example 1 (continued) Consider the recurrence operator:

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T and $\bar{T}$ are weakly and strongly desingularized operator (of order 3), respectively.

Contraction

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

Consider $\langle L \rangle := \mathbb{Q}(n)[\partial]L$, contraction of $\langle L \rangle$ to $\mathbb{Z}[n][\partial]$ is

$$\mathrm{Cont}(L) := \langle L \rangle \cap \mathbb{Z}[n][\partial]$$

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$$\mathrm{Cont}(L) := \langle L \rangle \cap \mathbb{Z}[n][\partial]$$

- $\mathrm{Cont}(L)$ is a finitely generated left ideal of $\mathbb{Z}[n][\partial]$.
- Every desingularized operator of L belongs to $\mathrm{Cont}(L)$.
- $\mathrm{Cont}(L)$ contains $\mathbb{Z}[n][\partial]L$, but in general more operators.
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Example 1 (continued) Consider the recurrence operator:

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$\mathrm{Cont}(L)$ is generated by $\{L, \bar{T}\}$.

Removability of polynomial factors

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

(Chen, Jaroschek, Kauers, Singer) Assume $p \mid \mathrm{lc}_\partial(L)$, p is primitive.

- If p is removable, then one can compute an upper bound k, such that there exists a p-removed operator T of order k.
- Using Euclidean algorithm, one can compute an upper bound for a weakly desingularized operator.

Removability of constant factors

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$. Write it as

$$L = a_0 f_0(n) + a_1 f_1(n)\partial + \cdots + a_m f_m(n)\partial^m$$

where $a_i \in \mathbb{Z}$, $f_i(n)$ is primitive.
If $\gcd(a_0, \ldots, a_m) = 1$, then we call L contant-primitive.

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Lemma (Gauss's Lemma for Ore Algebra) Suppose $L, P \in \mathbb{Z}[n][\partial]$. If L and P are constant-primitive, then PL is also constant-primitive.

Theorem 1 Suppose $L \in \mathbb{Z}[n][\partial]$ is constant-primitive, $a \in \mathbb{Z}$, $a \mid \mathrm{lc}_\partial(L)$. Then a is non-removable.

Removability of constant factors

Example 2 Consider

$$L = 3(n+2)(3n+4)(3n+5)(7n+3)\left(25n^2+21n+2\right)\partial^2 + \text{lower terms} \in \mathbb{Z}[n][\partial]$$

which is a constant-primitive recurrence operator for $a\binom{4n}{n} + b3^n$, where $a, b \in \mathbb{Z}$. From Theorem 1, 3 is non-removable.

Desingularization at a fixed order

Given $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$.

Question (A): Given a fixed order k, how to find a strongly desingularized operator T of order k?

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Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

Question (A): Given a fixed order k, how to find a strongly desingularized operator T of order k?
We define

$$
\begin{aligned}
M_k &:= \{T \mid T \in \mathrm{Cont}(L),\ \deg_\partial(T) \le k\} \\
I_k &:= \{\mathrm{lc}_\partial(T)(n-k) \mid T \in \mathrm{Cont}(L),\ \deg_\partial(T) = k\} \cup \{0\}
\end{aligned}
$$

If T is a strongly desingularized operator of order k, then $\mathrm{lc}_\partial(T)(n-k) \in I_k$. So, we consider

Question (B): Given a fixed order k, how to find a basis $\mathbf{b}$ of I_k and its corresponding operator $\mathbf{B}$ in M_k?

Syzygy

Let $V := \{v_1, \ldots, v_m\}$ be a finite set of $\mathbb{Z}[n]^r$.

We call the set $\{(a_1, \ldots, a_m) \in \mathbb{Z}[n]^m \mid \sum_{i=1}^m a_i \cdot v_i = 0\}$ the module of syzygies of V.

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Given $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$.

Theorem 2 For a fixed order k, one can compute a finite set $V \subseteq \mathbb{Z}[n]^r$ such that M_k is isomorphic to the module of syzygies of V as $\mathbb{Z}[n]$-module.

For $T = \sum_{i=0}^k c_i \partial^i \in \mathbb{Z}[n][\partial]$, we use $[\partial^i]T := c_i$ to refer the coefficient of ∂^i in T.

Proposition If $\mathbf{B} := \{B_1, \ldots, B_t\}$ is a basis of M_k, then $I_k = \langle ([\partial^k]B_1)(n-k), \ldots, ([\partial^k]B_t)(n-k) \rangle$.

An algorithm for desingularization

Algorithm 1 Input: $L \in \mathbb{Z}[n][\partial], \deg_\partial(L) = r$ and $k \geq r$. Output: a basis **b** of I_k, its corresponding operators **B** in M_k.

1. Compute $\operatorname{rrem}(\partial^j, L) := \sum_{i=1}^r u_{ij}\partial^{i-1}$, $0 \leq j \leq k$. Let $U := (u_{ij}) \in \mathbb{Q}(n)^{r\times(k+1)}$.
2. Compute $d_i :=$ the least common multiples of denominators of i-th row vector of U. Let $v_{ij} := d_i u_{ij}$, $1 \leq i \leq r, 0 \leq j \leq k$. Let $v_j := (v_{1j}, \ldots, v_{rj})^T \in \mathbb{Z}[n]^r$ and $V := \{v_0, \ldots, v_k\}$.
3. Compute a basis B of the module of syzygies of V.
4. Let $\mathbf{B} := \{\sum_{i=0}^k b_i\partial^i \mid (b_0, \ldots, b_k) \in B\}$ and $\mathbf{b} := \{([\partial^k]b)(n-k) \mid b \in \mathbf{B}\}$.
5. Output: **b** and **B**.

Example for desingularization

Example 3 Consider the recurrence operator:

$$L = (2n-1)(n-1)\partial^2 + \left(5n - 1 - 9n^2 + 2n^3\right)\partial + n(1+2n)$$

Using Algorithm 1, we find

$$I_3 = \langle 3, n-4 \rangle$$

The corresponding operators are:

$$\begin{aligned} F_1 &= 3\partial^3 + (20n-31)\partial^2 + \left(17n^2 - 76n + 43\right)\partial + 17n + 9 \\ F_2 &= (n-1)\partial^3 + (n-1)(4n-9)\partial^2 + \left(3n^3 - 19n^2 + 33n - 13\right)\partial \\ &\quad + 3n^2 - 4n - 3 \end{aligned}$$

Here, F_1 is a strongly desingularized operator for L of order 3.

Desingularization and Contraction

Question (C): Given $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$, how to compute a $\mathbb{Z}[n][\partial]$-basis of $\mathrm{Cont}(L) := \langle L \rangle \cap \mathbb{Z}[n][\partial]$?

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Idea: Find an order bound $k \geq r$, such that $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$.

Lemma 1 Let $L \in \mathbb{Z}[n][\partial]$, $\deg_\partial(L) = r$. Then:

- $\mathbb{Z}[n][\partial] \cdot M_k = \mathbb{Z}[n][\partial] \cdot M_{k+1}$ iff $I_k = I_{k+1}$ for each $k \geq r$.

From Lemma 1, if $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$, then $\{I_j\}_{j=k}^{\infty}$ is a stable chain.

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From Lemma 1, if $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$, then $\{I_j\}_{j=k}^{\infty}$ is a stable chain.

We can compute an order bound k, such that M_k contains a weakly desingularized operator T. However, this does not imply that $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_k$.

Desingularization and Contraction

Example 4 Consider the following recurrence operator (Kauers, Krattenthaler, Müller):

$$
\begin{aligned}
L \;=\; & (n+10)(n^6+47n^5+915n^4+9445n^3+54524n^2+166908n \\
& +211696)\partial^{10} + \text{ lower terms}
\end{aligned}
$$

We can get a weakly desingularized operator at order 11. Using Algorithm 1, we get the following table:

$$
\begin{aligned}
I_{11} &= \langle 11104n, 4n(n-466), n(n^2-34n+1336)\rangle \\
I_{12} &= \langle 4n, n(n-24)\rangle \\
I_{13} &= \langle 2n, n(n-26)\rangle \\
I_{14} &= \langle 1n\rangle \\
I_{15} &= \langle 1n\rangle \\
&\vdots
\end{aligned}
$$

Saturation

Example 4 (Continued) From Lemma 1, we can not conclude that $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_{11}$. We will show $\mathrm{Cont}(L) = \mathbb{Z}[n][\partial] \cdot M_{14}$.

Let I be a left ideal of $\mathbb{Z}[n][\partial]$, $a \in \mathbb{Z} \setminus \{0\}$, we call

$$I : a^{\infty} = \{ T \in \mathbb{Z}[n][\partial] \mid a^k T \in I, \text{for some } k \in \mathbb{N} \}$$

the saturation of I with respect to a.

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Theorem 3 Let $L \in \mathbb{Z}[n][\partial]$, $\deg_{\partial}(L) = r$. Suppose that M_k contains a weakly desingularized operator T, $\mathrm{lc}_{\partial}(T) = ag$, where $a \in \mathbb{Z}$, g is primitive. Then $\mathrm{Cont}(L) = (\mathbb{Z}[n][\partial] \cdot M_k) : a^{\infty}$.

An algorithm for contraction

Algorithm 2 Input: $L \in \mathbb{Z}[n][\partial]$. Output: a basis of $\mathrm{Cont}(L)$.

1. Derive an order bound k such that M_k contains a weakly desingularized operator.
2. Compute a basis of M_k and a weakly desingularized operator T by using Algorithm 1, where $\mathrm{lc}_\partial(T) = ag$, $a \in \mathbb{Z}$, g is primitive.
3. Compute a basis $\mathbf{G}$ of $(\mathbb{Z}[n][\partial] \cdot M_k) : a^\infty$ by using Gröbner bases. Output: $\mathbf{G}$

Example for contraction

Example 1 (continued) Consider the recurrence operator:

$$L = (1 + 16n)^2\partial^2 - (224 + 512n)\partial - (1 + n)(17 + 16n)^2$$

M_3 contains a weakly desingularized operator $\bar{T}$, such that $\mathrm{lc}_\partial(\bar{T}) = 1$. From Theorem 3,

$$\mathrm{Cont}(L) = (\mathbb{Z}[n][\partial] \cdot M_3) : 1^\infty = \mathbb{Z}[n][\partial] \cdot M_3.$$

By Algorithm 2, $\mathrm{Cont}(L)$ is generated by $\{L, \bar{T}\}$.

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Conjecture: Let $(a_n)_{\geq 0}$ and $(b_n)_{\geq 0}$ be two P-recursive sequences over the integers with leading coefficient n. Show that $(n! a_n b_n)_{\geq 0}$ is also a P-recursive sequence over the integers with leading coefficient n.

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Given two recurrence equations

$$\begin{aligned} na_n &= \alpha_1 a_{n-1} + \ldots + \alpha_s a_{n-s} \\ nb_n &= \beta_1 b_{n-1} + \ldots + \beta_t b_{n-t} \end{aligned}$$

We construct a minimal recurrence operator L for $c_n := n!a_n b_n$.
Task: Find a strongly desingularized operator for L.

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Algorithm 2 can be used to search for counterexamples. However, results of experiments suggest that this conjecture might be true!

Special cases

Case 1: Consider the recurrence equations:

$$\begin{aligned} na_n &= \alpha a_{n-1} \\ nb_n &= \beta_1 b_{n-1} + \ldots + \beta_t b_{n-t} \end{aligned}$$

where α, $\beta_i \in \mathbb{Z}[n]$. Then $c_n := n! a_n b_n$ satisfies

$$nc_n = \gamma_1 c_{n-1} + \ldots + \gamma_t c_{n-t}$$

where $\gamma_i := \beta_i \prod_{j=0}^{i-1} \alpha(n-j)$

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where α_i, β_j are parameters. Then $c_n := n! a_n b_n$ satisfies

$$nc_n = \gamma_1 c_{n-1} + \ldots + \gamma_9 c_{n-9}$$

where $\gamma_i \in \mathbb{Z}[\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, n]$.

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Thanks!