The Restriction Problem for D-finite Functions

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work in progress

Outline

- D-finite functions
- The restriction problem
- Three approaches
- Examples and discussions on Approach 2

Annihilating operators

Let $\mathbf{k}$ be a field of characteristic 0, and x, y two indeterminates.

Let $f = \exp\left(x^2 + y^2\right)$. We have

$$\frac{\partial}{\partial x} f \ - \ 2x \cdot f = 0, \qquad \frac{\partial}{\partial y} f \ - \ 2y \cdot f = 0.$$

Using $\mathbf{k}(x, y)[\partial_x, \partial_y]$ with $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$,

$$(\partial_x \ - \ 2x) \bullet f = 0, \qquad (\partial_y \ - \ 2y) \bullet f = 0.$$

L in $\mathbf{k}(x, y)[\partial_x, \partial_y]$ is called an annihilating operator of f if

$$L \bullet f = 0.$$

For example,

$$\partial_x \ - \ 2x, \qquad \partial_y \ - \ 2y$$

are two annihilating operators of f.

D-finite Functions

Let $R = \mathbf{k}(x, y)[\partial_x, \partial_y]$ be the ring of differential operators.

- the annihilator of f is $\operatorname{ann}(f) := \{L \in R : L \bullet f = 0\}$.
- A left ideal $I \subseteq R$ is D-finite if $\dim_{\mathbf{k}(x,y)}(R/I) < +\infty$.
- A function f is D-finite if $\operatorname{ann}(f)$ is D-finite.

Example (continued) Let $f = \exp\left(x^2 + y^2\right)$.

$$\operatorname{ann}(f) = \langle \partial_x - 2x,\ \partial_y - 2y \rangle,$$

$$\dim_{\mathbf{k}(x,y)}(R/\operatorname{ann}(f)) = 1 \quad \Rightarrow \quad f \text{ is D-finite}$$

Motivation

Let $f(x,y)$ be D-finite

Given: a basis $\{P_1,\ldots,P_s\}$ of $I \subseteq \operatorname{ann}(f)$ s.t. I is D-finite

Goal: Find a nonzero operator $P \in \mathbf{k}[x][\partial_x]$ such that

$$P \bullet (f(x,y)|_{y=0}) = 0 \qquad (*)$$

Idea: Compute a nonzero operator $P \in \mathbf{k}[x][\partial_x]$ such that

$$P + yQ \in \operatorname{ann}(f) \quad \text{for some } Q \in \mathbf{k}(x,y)[\partial_x,\partial_y] \qquad (**)$$

A sufficient condition:

- $(Q \bullet f(x,y))\,|_{y=0}$ is well-defined.
- $(P \bullet f(x,y))\,|_{y=0} = P \bullet (f(x,y)|_{y=0})$

then $(**) \Rightarrow (*)$

Restriction Problem

Let $f(x, y)$ be D-finite

Given: a basis $\{P_1, \ldots, P_s\}$ of $I \subseteq \operatorname{ann}(f)$ s.t. I is D-finite
Find: a nonzero operator $P \in \mathbf{k}[x][\partial_x]$ such that

$$P + yQ \in I \subseteq \operatorname{ann}(f) \quad \text{for some suitable } Q.$$

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Example Let $f = \frac{\sin(x-y)}{x-y}$.

By `Annihilator` in the `HolonomicFunctions` package,

$$\text{ann}(f) = \langle \partial_x + \partial_y, (x-y)\partial_y^2 - 2\partial_y + (x-y) \rangle$$

We can find $P = x\partial_x^2 + 2\partial_x + x$, $Q = \frac{2}{x-y}\partial_y$ such that

$$P + yQ \in \text{ann}(f).$$

Then

$$P \bullet (f(x, y)|_{y=0}) = 0.$$

Approach 1: Pure Elimination

Let $f(x, y)$ be D-finite.
Suppose G is a Gröbner basis of a D-finite left ideal $I \subseteq \text{ann}(f)$ w.r.t. a term order $\prec$.

Each $g \in \mathbf{k}(x, y)[\partial_x, \partial_y]$ can be reduced to a unique element by G, denoted as $\text{nf}(g, I)$.

Find a linear dependence over $\mathbf{k}(x, y)$ among

$$\begin{array}{l} \text{nf}(1, I) \\ \text{nf}(\partial_x, I) \\ \text{nf}(\partial_x^2, I) \\ \dots \end{array}$$

The result is a nonzero operator $T \in \mathbf{k}[x, y][\partial_x]$. Then

$$T|_{y=0} + yQ \in I \quad \text{for some} \quad Q.$$

Approach 2

Let $f(x, y)$ be D-finite

Find a linear dependence over $\mathbf{k}(x)$ among

$$\begin{array}{l} \mathrm{nf}(1, I)|_{y=0} \\ \mathrm{nf}(\partial_x, I)|_{y=0} \\ \mathrm{nf}(\partial_x^2, I)|_{y=0} \\ \ldots \end{array}$$

Suppose that the result is $P \in \mathbf{k}[x][\partial_x]$. Then

$$P + yQ \in I \quad \text{for some} \quad Q$$

Approach 2

Let $f(x,y)$ be D-finite

Find a linear dependence over $\mathbf{k}(x)$ among

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Suppose that the result is $P \in \mathbf{k}[x][\partial_x]$. Then

$$P + yQ \in I \quad \text{for some} \quad Q$$

Note: Sometimes we can not get P

Approach 3: Algorithm [Saito and Takayama (1994)]

Let $f(x, y)$ be D-finite

Let $J = \{L \in \mathbf{k}[x, y][\partial_x, \partial_y] \mid L \bullet f = 0\}$

Find a linear dependence over $\mathbf{k}(x)$ among

$$\begin{array}{l} \mathrm{nf}(1, J|_{y=0}) \\ \mathrm{nf}(\partial_x, J|_{y=0}) \\ \mathrm{nf}(\partial_x^2, J|_{y=0}) \\ \dots \end{array}$$

The result is $P \in \mathbf{k}[x][\partial_x]$. Then

$$P + yQ \in J \quad \text{for some} \quad Q$$

Note: $J|_{y=0}$ is a $\mathbf{k}[x][\partial_x]$-submodule of $\mathbf{k}[x][\partial_x, \partial_y]$

Comparison

Approach 1	Approach 2	Approach 3
I	I	J
$\downarrow$ nf	$\downarrow$ nf	$\downarrow$ $y = 0$
$\downarrow$ find rel	$\downarrow$ $y = 0$	$\downarrow$ nf
$\downarrow$ $y = 0$	$\downarrow$ find rel	$\downarrow$ find rel
$P + yQ \in I$	$P + yQ \in I$	$P + yQ \in J$
naive	semi-clever	clever

Example

Approach 1: Let $f = \frac{\sin(x-y)}{x-y}$.

By `Annihilator` in the `HolonomicFunctions` package,

$$\text{ann}(f) = \langle \partial_x + \partial_y, (x-y)\partial_y^2 - 2\partial_y + (x-y) \rangle$$

Find a linear dependence over $\mathbf{k}(x, y)$ among

$$\begin{aligned} &\text{nf}(1, \text{ann}(f)) = 1 \\ &\text{nf}(\partial_x, \text{ann}(f)) = -\partial_y \\ &\text{nf}(\partial_x^2, \text{ann}(f)) = \tfrac{2}{x-y}\partial_y - 1 \\ &\dots \end{aligned}$$

The result is $T = (x-y)\partial_x^2 + 2\partial_x + (x-y)$. Then,

$$T|_{y=0} + yQ \in \text{ann}(f) \quad \text{for some} \quad Q$$

Example (continued)

Approach 2: Let $f = \frac{\sin(x-y)}{x-y}$.

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$$\mathrm{ann}(f) = \langle \partial_x + \partial_y, (x-y)\partial_y^2 - 2\partial_y + (x-y) \rangle$$

Find a linear dependence over $\mathbf{k}(x)$ among

$$\begin{aligned}
&\mathrm{nf}(1, \mathrm{ann}(f))|_{y=0} = 1\\
&\mathrm{nf}(\partial_x, \mathrm{ann}(f))|_{y=0} = -\partial_y\\
&\mathrm{nf}(\partial_x^2, \mathrm{ann}(f))|_{y=0} = \frac{2}{x}\partial_y - 1\\
&\dots
\end{aligned}$$

The result is $P = x\partial_x^2 + 2\partial_x + x$.

Example (continued)

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&\dots
\end{aligned}$$

The result is $P = x\partial_x^2 + 2\partial_x + x$.

Note that in this case

$$P = T|_{y=0}.$$

Example 2

Approach 1: Let $f = \sin(x^2 + y^2)$.

By `Annihilator` in the `HolonomicFunctions` package,

$$\mathrm{ann}(f) = \langle y\partial_x - x\partial_y, -y\partial_y^2 + \partial_y - 4y^3 \rangle$$

Find a linear dependence over $\mathbf{k}(x, y)$ among

$$\begin{aligned} &\mathrm{nf}(1, \mathrm{ann}(f)) = 1 \\ &\mathrm{nf}(\partial_x, \mathrm{ann}(f)) = \tfrac{x}{y}\partial_y \\ &\mathrm{nf}(\partial_x^2, \mathrm{ann}(f)) = \tfrac{1}{y}\partial_y - 4x^2 \\ &\ldots \end{aligned}$$

The result is $T = x\partial_x^2 - \partial_x + 4x^3$. Then

$$T|_{y=0} + yQ \in \mathrm{ann}(f) \quad \text{for some} \quad Q$$

Example 2 (continued)

Approach 2: Let $f = \sin(x^2 + y^2)$.

By `Annihilator` in the `HolonomicFunctions` package,

$$\text{ann}(f) = \langle y\partial_x - x\partial_y, -y\partial_y^2 + \partial_y - 4y^3 \rangle$$

Find a linear dependence over $\mathbf{k}(x)$ among

$$\begin{aligned}
&\text{nf}(1, \text{ann}(f))|_{y=0} = 1 \\
&\text{nf}(\partial_x, \text{ann}(f))|_{y=0} = \left(\tfrac{x}{y}\partial_y\right)|_{y=0} \\
&\text{nf}(\partial_x^2, \text{ann}(f))|_{y=0} = \left(\tfrac{1}{y}\partial_y - 4x^2\right)|_{y=0} \\
&\ldots
\end{aligned}$$

Approach 2 fails!

Example 2 (continued)

A modified Approach 2: Let $f = \sin(x^2 + y^2)$.

By `Annihilator` in the `HolonomicFunctions` package,

$$\operatorname{ann}(f) = \langle y\partial_x - x\partial_y, -y\partial_y^2 + \partial_y - 4y^3 \rangle$$

Find a linear dependence over $\mathbf{k}(x)$ among

$$\begin{aligned} &\operatorname{nf}(1, \operatorname{ann}(f)) = 1 \\ &\operatorname{nf}(\partial_x, \operatorname{ann}(f)) = \left(\tfrac{x}{y}\partial_y\right) \\ &\operatorname{nf}(\partial_x^2, \operatorname{ann}(f)) = \left(\tfrac{1}{y}\partial_y - 4x^2\right) \\ &\ldots \end{aligned}$$

Example 2 (continued)

Let

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} \frac{x}{y} \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{y} \\ -4x^2 \end{pmatrix}$$

By Cramer's rule, Let

$$\Delta_0 = \begin{vmatrix} \frac{1}{y} & \frac{x}{y} \\ -4x^2 & 0 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} 0 & \frac{1}{y} \\ 1 & -4x^2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 0 & \frac{x}{y} \\ 1 & 0 \end{vmatrix}$$

we have

$$-\Delta_2 \cdot \mathbf{v}_2 + \Delta_1 \cdot \mathbf{v}_1 + \Delta_0 \cdot \mathbf{v}_0 = 0$$

By clearing the denominator, setting $y = 0$ and expanding the determinants,

$$x \cdot \mathbf{v}_2 - \mathbf{v}_1 + 4x^3 \cdot \mathbf{v}_0 = 0$$

Get

$$P = x\partial_x^2 - \partial_x + 4x^3$$

Problem

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Question 2: What are the necessary and sufficient conditions for:

$$P + yQ \in \mathrm{ann}(f) \quad \text{for some } Q \in \mathbf{k}(x,y)[\partial_x, \partial_y]$$

$$\Updownarrow$$

$$P \bullet (f(x,y)|_{y=0}) = 0$$

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Question 1: To what extent does the modified Approach 2 work?

Question 2: What are the necessary and sufficient conditions for:

$$P + yQ \in \mathrm{ann}(f) \quad \text{for some } Q \in \mathbf{k}(x, y)[\partial_x, \partial_y]$$

$$\Updownarrow$$

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Thanks!