On Sequences Associated to the Invariant Theory of Rank Two Lie Algebras

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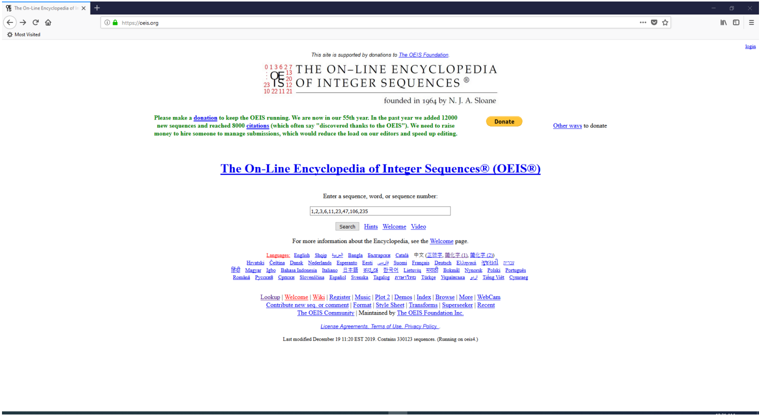
The On-Line Encyclopedia of Integer Sequences (OEIS)



OEIS is an online database of integer sequences, such as Fibonacci numbers (A000045), Catalan numbers (A000108).

Two families of sequences in OEIS

a (OEIS tag)	0	1	2	3	4	5
A059710	1	0	1	1	4	10
A108307	1	1	2	5	15	51
A108304	1	2	5	15	52	202

The first family of sequences (octant sequences)

a (OEIS tag)	0	1	2	3	4	5
A151366	1	0	2	2	12	30
A236408	1	1	3	9	33	131
A001181	1	2	6	22	92	422
A216947	1	3	11	49	221	1113

The second family of sequences (quadrant sequences)

- Those sequences are associated to the invariant theory of the exceptional simple Lie algebra G_2 of rank 2.
- The quadrant sequences are related to the octant sequences by the branching rules for $SL(3)$ of G_2.

Octant sequences

The first family of sequences can be interpreted as lattice walks restricted to the octant. We call them octant sequences.

- A059710: enumerates the multiplicities of the trivial representation in the tensor powers of V, which is the 7-D fundamental representation of G_2.
- A108307: enumerates enhanced 3-noncrossing set partitions.
- A108304: enumerates 3-noncrossing set partitions.

(Lin, 2018; Gil and Tirrell, 2019): A108307 and A108304 are related by the binomial transform.

Motivation and Contribution

(Bostan, Tirrell, Westbury and Z., 2019): A059710 and A108307 are also related by the binomial transform.

Mihailovs' conjecture: Let $T_3(n)$ be the n-th term of A059710. Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1)$$
$$+2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

(Bostan, Tirrell, Westbury and Z., 2019): Three independent proofs of Mihailovs' conjecture.

- Two proofs are based on binomial relation between A059710 and A108307, together with a result by Bousquet-Mélou and Xin.
- The third one is a direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of T_3 in terms of hypergeometric functions.

Quadrant sequences

The second family of sequences can be interpreted as lattice walks restricted to the quadrant. We call them quadrant sequences.

- A151366: enumerates nonpositive bipartite trivalent graphs.
- A236408: enumerates pasting diagrams.
- A001181: enumerates Baxter permutations.
- A216947: enumerates 2-coloured noncrossing set partitions.

Question: What are relations between quadrant sequences?

Motivation and Contribution

(Marberg, 2013): a combinatorial proof that A151366, A001181, and A216947 are related by binomial transforms.

(Bostan, Tirrell, Westbury and Z., 2019): Derive a uniform recurrence equation for quadrant sequences and show that they are related by binomial transform by the representation theory of simple Lie algebras.

Outline

- binomial relation between the first and second octant sequences
- Three independent proofs of Mihailovs' conjecture
- Recurrence relations for the quadrant sequences

Preliminaries

Definition 1 Let G be a reductive complex algebraic group and let V be a representation of G. The sequence associated to (G, V), denoted $\mathbf{a}_V$, is the sequence whose n-th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.

Example 1 Let V be the 7-D fundamental representation of G_2. Then A059710 is the sequence associated with (G_2, V).

Let $\mathbf{a}$ be a sequence with n-th term $a(n)$, the binomial transform of $\mathbf{a}$ is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} a(i).$$

Preliminaries

Lemma 1 Assume $\mathbf{a}_V$ is the sequence associated to (G, V) as specified in Definition 1. Then $\mathbf{a}_{V\oplus\mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Lemma 2 Assume $\mathbf{a}$ enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \coprod \{0\}$.

Lemma 3 Let $G(t)$ be the generating function of $\mathbf{a}$. For $k \in \mathbb{Z}$, denote the generating function of $\mathcal{B}^k\mathbf{a}$ by $\mathcal{B}^k G$. Then

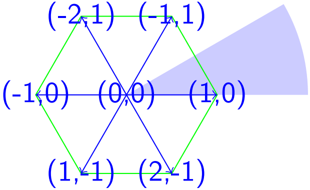
$$(\mathcal{B}^k G)(t) = \frac{1}{1-k\ t} G\left(\frac{t}{1-k\ t}\right).$$

Binomial relation between A059710 and A108307

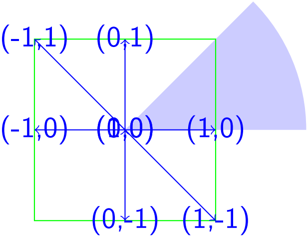
Let V be the 7-D fundamental representation of G_2. Then

- A059710 is the sequence associated to (G_2, V). Let $T_3(n)$ be its n-th term.
- A108307 enumerates enhanced 3-noncrossing set partitions. Let $E_3(n)$ be its n-th term.

In terms of lattice walks, we can interpret T_3 and E_3 as follows:



Steps in weight lattice of G_2



Steps in octant related to $E_3(n)$

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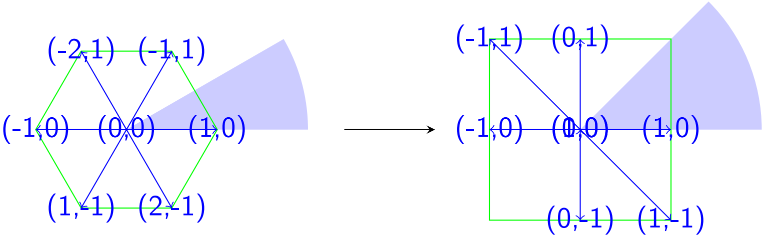

Steps in weight lattice of G_2

Steps in octant related to $E_3(n)$

If we make a linear transformation $(x, y) \to (x + y, y)$, then it identifies the six non-zero steps, as well as the two domains.

Binomial relation between A059710 and A108307

Recall: Lemma 2 Assume **a** enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \coprod \{0\}$.

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By Lemma 2 and the previous figures, we conclude that E_3 is the binomial transform of T_3.

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Recall: Lemma 1 Assume $\mathbf{a}_V$ is the sequence associated to (G, V) as specified in Definition 1. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Thus, the octant sequences are sequences associated to

$$(G_2, V), \quad (G_2, V \oplus \mathbb{C}), \quad (G_2, V \oplus 2\mathbb{C}).$$

First proof of Mihailovs' conjecture

Mihailovs' conjecture: Let $T_3(n)$ be the n-th term of A059710. Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)\,T_3(n) + (n+2)(19n+75)\,T_3(n+1) \\ +2(n+2)(2n+11)\,T_3(n+2) - (n+8)(n+9)\,T_3(n+3) = 0.$$

(Bousquet-Mélou and Xin, 2005): Let $E_3(n)$ be the n-th term of A108307. Then E_3 is given by $E_3(0) = E_3(1) = 1$, and

$$8(n+3)(n+1)\,E_3(n) + \left(7n^2 + 53n + 88\right) E_3(n+1) \\ - (n+8)(n+7)\,E_3(n+2) = 0.$$

First proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3. Thus,

$$T_3(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n,k) = (-1)^{n-k}\binom{n}{k}E_3(k)$.

- By Bousquet-Mélou and Xin's result, $f(n,k)$ is holonomic function, which satisfies ordinary difference equations for n and k, respectively.
- Idea: Using creative telescoping method (Zeilberger, 1990), which is an algorithmic approach to compute a differential/difference equation for the integration/sum of holonomic functions, to drive a recurrence equation for T_3.

First proof of Mihailovs' conjecture

- Using the Koutschan's Mathematica package `HolonomicFunctions.m` that implements Chyzak's algorithm for creative telescoping, we find exactly the recurrence equation in Mihailovs' conjecture.

Creative Telescoping

Prove

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

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Set $f(n,k) = \binom{n}{k}$ and $F(n) = \sum_{k=0}^{n} \binom{n}{k}$. Find

$$1 \cdot f(n+1,k) + (-2) \cdot f(n,k) = \Delta_k \left(\frac{k}{k-n-1} \cdot f(n,k) \right) \quad (1)$$

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Taking sums on both sides of (1) for k from $-\infty$ to ∞, we get

$$\sum_{k=0}^{n+1} f(n+1,k) - 2\sum_{k=0}^{n} f(n,k) = 0$$

because $f(n,k) = 0$ if $k < 0$ or $k > n$. Thus, we have

Creative Telescoping

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$$F(n+1) - 2F(n) = 0.$$

Together with $F(0) = 1$, we get $F(n) = 2^n$.

Second proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3. Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$ and $\mathcal{E}(t) = \sum_{n\geq 0} E_3(n)t^n$. Then

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

- By Bousquet-Mélou and Xin's result, we can derive an ODE for $\mathcal{E}(t)$.
- Using the closure properties of holonomic function (the sum, product and algebraic substitution of holonomic functions is still holonomic), we can derive an ODE for $\mathcal{T}(t)$ and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence equation in Mihailovs' conjecture.

Third proof of Mihailovs' conjecture

Idea: In terms of lattice walks, we can interpret $T_3(n)$ to be the constant term of $W\,K^n$, where

$$K = (1 + x + y + x\,y + x^{-1} + y^{-1} + (xy)^{-1}),$$

and

$$W = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$. Then $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1 - tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(xy - txyK)$, which is proportional to the contour integral of $W/(xy - txyK)$ over a cycle.

Third proof of Mihailovs' conjecture

Using creative telescoping method, we can compute a 6-th order ODE for $\mathcal{T}(t)$. Moreover, by using factorization of differential operators, we can show that $L_3(\mathcal{T}(t)) = 0$, where $\partial = \frac{d}{dt}$ and

$$L_3 = t^2 (2t+1)(7t-1)(t+1)\partial^3 + 2t(t+1)\left(63t^2+22t-7\right)\partial^2 + \left(252t^3+338t^2+36t-42\right)\partial + 28t(3t+4).$$

Converting it into a linear recurrence for $T_3(n)$, we get exactly the recurrence equation in Mihailovs' conjecture.

Closed formulae

By factorization of the operator L_3 and algorithms for solving 2-nd order ODEs, we derive the following closed formula for $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \frac{1}{30\,t^5}\left[R_1 \cdot {}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 2\end{matrix}; \phi\right) + R_2 \cdot {}_2F_1\left(\begin{matrix}\frac{2}{3} & \frac{4}{3} \\ & 3\end{matrix}; \phi\right) + 5\,P\right],$$

where

$$R_1 = \frac{(t+1)^2\left(214\,t^3 + 45\,t^2 + 60\,t + 5\right)}{t-1},$$

$$R_2 = 6\,\frac{t^2\,(t+1)^2\left(101\,t^2 + 74\,t + 5\right)}{(t-1)^2},$$

and

$$\phi = \frac{27\,(t+1)\,t^2}{(1-t)^3}, \qquad P = 28\,t^4 + 66\,t^3 + 46\,t^2 + 15\,t + 1.$$

Closed formulae

By elliptic curve theory, we derive an alternative formula for $\mathcal{T}(t)$:

$$\frac{P}{6\,t^5} + \frac{(7\,t-1)\,(2\,t+1)\,(t+1)}{360\,t^5}\Big(\left(155\,t^2+182\,t+59\right)(11\,t+1)\,H(t) \\ + \left(341\,t^3+507\,t^2+231\,t+1\right)(5\,t+1)\,H'(t)\Big),$$

where

$$H(t) = \frac{1}{{g_2}^{1/4}} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{12} & \frac{5}{12} \\ & 1\end{matrix}; \frac{1728}{J}\right),$$

$$J = \frac{(t-1)^3\left(25\,t^3+21\,t^2+3\,t-1\right)^3}{t^6\,(1-7\,t)\,(2\,t+1)^2\,(t+1)^3},$$

and

$$g_2 = (t-1)\left(25\,t^3+21\,t^2+3\,t-1\right).$$

Transcendence and asymptotics

Using those closed formulae, we can show that that $\mathcal{T}(t)$ is a transcendental power series and its n-th coefficient

$$T_3(n) \sim C \cdot \frac{7^n}{n}, \quad \text{where } C = \frac{4117715}{864}\frac{\sqrt{3}}{\pi} \approx 2627.6.$$

Recurrence relations for quadrant sequences

Definition 2 Let $\tilde{V}$ be the defining representation of $SL(3)$ and denote the dual by $\tilde{V}^*$. For $k \geqslant 0$, we define $\mathcal{S}_k$ to be the sequence associated to $(SL(3), \tilde{V} \oplus \tilde{V}^* \oplus k\,\mathbb{C})$.

Remark: $SL(3)$ is the maximal subgroup of G_2. Let V be the 7-D fundamental representation of G_2. Then $\mathcal{S}_k$ is the the sequence associated to $(SL(3), (V \oplus k\mathbb{C}) \downarrow_{SL(3)})$.

Theorem (Bostan, Tirrell, Westbury and Z., 2019): The quadrant sequences $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to the sequences in the second family listed in OEIS.

Lemma 4 Let $\mathcal{G}_k$ be the generating function of $\mathcal{S}_k$, where $k \geq 0$. Then $\mathcal{G}_k$ is the constant coefficient of $[x^0y^0]$ of $W/(1-tK)$, where

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x}$$

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2y^2 + y^3 - \frac{y^2}{x}.$$

Recurrence relations for quadrant sequences

By Lemma 4, $\mathcal{S}_3$ is identical to the sequence A216947.

(Marberg, 2013): The n-th term $C_2(n)$ of $\mathcal{S}_3$ is given by $C_2(0) = 1, C_2(1) = 3$ and

$$\begin{aligned}(n+5)(n+6)\cdot C_2(n+2) - 2(5n^2+36n+61)\cdot C_2(n+1)\\ + 9(n+1)(n+4)\cdot C_2(n) = 0.\end{aligned}$$

By Lemma 1, $\mathcal{S}_k$'s are related by binomial transforms. Thus, by Lemma 3, the generating function of $\mathcal{S}_k$ is

$$\mathcal{G}_k(t) = \frac{1}{1-kt}\cdot \mathcal{G}_3\left(\frac{t}{1-kt}\right)$$

where $\mathcal{G}_3(t)$ is the generating function of $\mathcal{S}_3$.

Recurrence relations for quadrant sequences

Using closure properties of holonomic functions, we derive a uniform 4-th order recurrence equation for $\mathcal{S}_k$ with k as a parameter.

By comparing the recurrence equations between $\mathcal{S}_k$'s and the sequences in the second family, and then checking initial terms, we show that

Corollary: The recurrence relations stated in OEIS for the sequences in the second family are true.

Summary

- A combinatorial proof of binomial relation between the first and second octant sequences
- Three independent proofs of Mihailovs' conjecture
  - Two proofs are based on binomial relation between the first and second octant sequences
  - A direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of the first octant sequence
- A unified proof for recurrence relations of the quadrant sequences

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Thanks!