Convergent expansions and bounds for the incomplete elliptic integral of the second kind near the logarithmic singularity

Dmitrii Karp [*], Yi Zhang [†]

May 29, 2023

Abstract

We find two series expansions for Legendre's second incomplete elliptic integral $E(\lambda, k)$ in terms of recursively computed elementary functions. Both expansions converge at every point of the unit square in the (λ, k) plane. Partial sums of the proposed expansions form a sequence of approximations to $E(\lambda, k)$ which are asymptotic when λ and/or k tend to unity, including when both approach the logarithmic singularity $\lambda = k = 1$ from any direction. Explicit two-sided error bounds are given at each approximation order. These bounds yield a sequence of increasingly precise asymptotically correct two-sided inequalities for $E(\lambda, k)$. For the reader's convenience we further present explicit expressions for low-order approximations and numerical examples to illustrate their accuracy. Our derivations are based on series rearrangements, hypergeometric summation algorithms and extensive use of the properties of the generalized hypergeometric functions including some recent inequalities.

Keywords: *Legendre's elliptic integrals, incomplete elliptic integral of the second kind, asymptotic approximation, two-sided bounds, hypergeometric function, symbolic computation, symmetric elliptic integrals*

MSC2020: 33E05, 33F10, 33C20, 33C60,

1 Introduction

Legendre's second elliptic integral (EI) is defined by [5, (2.2)]

$$E(\lambda, k) = \int_0^\lambda \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt. \tag{1}$$

It can be expressed in terms of Appell's hypergeometric function [5, (2.7)]

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!}$$

[*] Holon Institute of Technology, Holon, Israel. Email: dimkrp@gmail.com

[†] Corresponding author. Department of Foundational Mathematics, School of Science, Xi'an Jiaotong-Liverpool University, Suzhou, 215123, China. The work of Y. Zhang was supported by XJTLU Research Development Fund No. RDF-20-01-12, the NSFC Young Scientist Fund No. 12101506 and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China No. 21KJB110032. Email: Yi.Zhang03@xjtlu.edu.cn

as follows [5, (2.9)]

$$E(\lambda,k)=\lambda F_1(1/2;1/2,-1/2;3/2;\lambda^2,k^2\lambda^2). \tag{2}$$

The double series defining F_1 converges in the domain $|\lambda^2|<1$, $|k^2\lambda^2|<1$ in the space $\mathbb{C}^2$ of the complex variables (λ,k) and defines an analytic function there. Clearly, the bi-disk $|\lambda|<1$, $|k|<1$ is (properly) contained in the convergence domain. The function F_1 in (2) can be analytically continued to the domain

$$\left\{|k|<1,|k\lambda|>1,|\arg(-\lambda^2)|<\pi,|\arg(-k^2\lambda^2)|<\pi\right\}$$

according to [3, Proposition 5] and further to $|k|>1$, $|\lambda|>1$ and the same restrictions on the arguments via the reflection relation [21, (19.7.4)]

$$kE(\lambda,1/k)=E(\lambda/k,k)-(1-k^2)F(\lambda/k,k),$$

where

$$F(\lambda,k)=\int_0^\lambda\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\lambda F_1(1/2;1/2,1/2;3/2;\lambda^2,k^2\lambda^2)$$

is the first Legendre's incomplete elliptic integral [5, (2.8)]. Note that $E(\lambda,1)=\lambda$ for each $0<\lambda<1$, while $E(1,k)=E(k)$, which is the complete elliptic integral of the second kind. Expansions for $F(\lambda,k)$ analogous to those derived in this paper for $E(\lambda,k)$ were found by the first author jointly with S.M. Sitnik in [15].

The following two symmetric standard EIs are defined in [5, 6, 9, 10] as follows:

$$R_F(x,y,z)=\frac{1}{2}\int_0^\infty\frac{dt}{\sqrt{(t+x)(t+y)(t+z)}},$$
$$R_D(x,y,z)=\frac{3}{2}\int_0^\infty\frac{dt}{(t+z)\sqrt{(t+x)(t+y)(t+z)}},$$

and related to $E(\lambda,k)$ by [7, (4.2)]

$$E(\lambda,k)=\lambda R_F(1-\lambda^2,1-k^2\lambda^2,1)-\frac{1}{3}k^2\lambda^3R_D(1-\lambda^2,1-k^2\lambda^2,1). \tag{3}$$

Asymptotic expansions for $E(\lambda,k)$ near the point $(1,1)$ appeared in [12, 13]. For symmetric elliptic integrals with one of the parameters going to infinity, the first (and the second in some cases) term of the asymptotic expansion of R_F, R_D, and R_J, as well as a quite accurate bounds for the remainder, have been obtained by Carlson and Gustafson [10]. Moreover, for all the symmetric EIs, they also considered the case of several parameters going to infinity. The first approximation of Carlson and Gustafson has been extended to the general zero-balanced Appell function F_1 by the first author in [14]. Complete convergent expansions for symmetric EIs (and not only first terms) have been obtained earlier by Carlson using Mellin transform techniques [8], but computation of the higher order terms is not at all straightforward and the error bounds are not satisfactory [8, Section 3]. The complete asymptotic expansions with recursively computed terms and explicit error bounds at each approximation order were obtained by López in [18, 19] for various asymptotic regimes. Formula (3) allows converting his results into the asymptotic approximations for Legendre's EI $E(\lambda,k)$ as $\lambda\to1$ (while k is fixed or tends to 1 as well). Details of these conversion are given in the Appendix to this paper. The resulting approximation takes the form

$$E(\lambda,k)=\lambda(1-k^2\lambda^2)\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}}+k^2\lambda^3+r_1, \tag{4}$$

with the remainder r_1 satisfying (72) and (73). For a more comprehensive overview of the theories, algorithms, and applications of elliptic integrals, we refer to [1, 2].

In this paper we will derive two types of convergent series directly for $E(\lambda, k)$ which are also asymptotic when either λ or k or both tend to 1. Each of the two series converges at each point of the unit square in the (λ, k) plane. Convergence is uniform on compact subsets of the closed unit square with one boundary segment removed ($\lambda = 1$ ($k = 1$) for the first (second) expansion). We further furnish explicit two-sided error bounds at each approximation order. Hence, our results can also be interpreted as a sequence of asymptotically precise (as $\lambda, k \to 1$) two-sided inequalities for the second incomplete EI $E(\lambda, k)$. Our derivation does not rely on asymptotic methods and uses standard analytic techniques combined with the algorithms of symbolic computation and some recent and rather accurate inequalities for the generalized hypergeometric function. This leads to high-precision approximations which are much better than those present in the literature so far. We demonstrate this numerically in the ultimate section of the paper. For example, our first order approximation is given by

$$E_1(\lambda, k) = (\lambda - 1/\lambda)\sqrt{1 + (\lambda^2(1-k^2))/(1-\lambda^2)} - \frac{1-k^2}{4}\ln\frac{1-\lambda}{1+\lambda} + 1/\lambda.$$

This approximation is also an upper bound. We further propose a sequence of more precise refined approximations which do not constitute (neither upper nor lower) bounds. For instance, the first order refined approximation of the first kind is given by

$$\begin{aligned}\hat{E}_1(\lambda, k) = (\lambda - 1/\lambda)\sqrt{1 + \frac{\lambda^2(1-k^2)}{1-\lambda^2}} - \frac{(101+19k^2)(1-k^2)}{32(7+8k^2)}\ln\frac{1-\lambda}{1+\lambda} + 1/\lambda \\ - \frac{675\sqrt{2}(1-k^2)^{3/2}}{128(7+8k^2)\sqrt{15-7\lambda^2-8\lambda^2k^2}}\ln\frac{\sqrt{15-7\lambda^2-8\lambda^2k^2}+\lambda\sqrt{8(1-k^2)}}{\sqrt{15-7\lambda^2-8\lambda^2k^2}-\lambda\sqrt{8(1-k^2)}}.\end{aligned}$$

Table 1 in Section 5 shows a remarkable accuracy of this approximation.

The paper is organized as follows. In Section 2 succeeding this introduction we rederive two known series expansions for $E(\lambda, k)$ using partial fractions and the generating function for Legendre's polynomials and find new bounds for the remainders. These expansions then serve as the starting points for new expansions established in Sections 3 and 4. Both of them converge for any fixed $(\lambda, k) \in (0, 1)\times(0, 1)$. The partial sums of the first expansion derived in Section 3 form an asymptotic series as $k \to 1$ which is uniform with respect to λ lying in any subset of the unit square with bounded ratio $(1-k)/(1-\lambda)$. In a similar fashion, the partial sums of the second expansion derived in Section 4 form an asymptotic series as $\lambda \to 1$ which is uniform with respect to k lying in any subset of the unit square with bounded ratio $(1-\lambda)/(1-k)$. In Section 5 we present the results of numerical experiments illustrating high precision of our first and second approximations and even more so for the refined approximations obtained by incorporating the error bounds into the formulas. Finally, we included a short appendix containing a conversion of the first approximations for the symmetric elliptic integrals due to Carlson-Gustafson [10] and López [18] and their error bounds into the corresponding results for the incomplete Legendre's second elliptic integral.

2 Expansions of Byrd-Friedman and Carlson revisited

In this section we present two auxiliary series expansions which will be the cornerstones for the main results given in Sections 3 and 4. The first expansion can be regarded as an equivalent form of a known expansion due to Byrd-Friedman [4], while the second one is derived from an expansion by Bille C. Carlson [5] via certain hypergeometric transformations. The error bounds given in this section appear to be new.

To deduce the first expansion, we need the following lemma.

Lemma 2.1. *For an integer $j \geq 1$ and $0 \leq \lambda < 1$, we have the following identity:*

$$\int_0^\lambda \frac{t^{2j}dt}{(1-t^2)^j} = \frac{\lambda^{2j+1}}{2j+1} {}_2F_1(j, j+1/2; j+3/2; \lambda^2)$$
$$= \frac{\lambda^{2j+1}}{(1-\lambda^2)^j} + (-1)^j \frac{(1/2)_j}{(j-1)!} \ln \frac{1-\lambda}{1+\lambda} + \frac{1}{\lambda} \sum_{n=0}^{j-1} (-1)^{n-1} \frac{(1/2-j)_n}{(1-j)_n} \left(\frac{\lambda^2}{1-\lambda^2} \right)^{j-n}, \quad (5)$$

where

$$(b)_0 = 1, \quad (b)_n = b(b+1)(b+2)\cdots(b+n-1),\ n \geq 1,$$

is the Pochhammer symbol (or the rising factorial).

Proof. The first equality is the direct consequence of Euler's integral representation [21, 15.6.1]. To establish the second equality by using integration by parts, we have

$$I = \int_0^\lambda \frac{t^{2j}dt}{(1-t^2)^j} = \frac{1}{2j+1} \int_0^\lambda \frac{d(t^{2j+1})}{(1-t^2)^j}$$
$$= \frac{1}{2j+1} \left[\frac{t^{2j+1}}{(1-t^2)^j} \Big|_0^\lambda - \int_0^\lambda t^{2j+1} d\left(\frac{1}{(1-t^2)^j} \right) \right]$$
$$= \frac{1}{2j+1} \cdot \frac{\lambda^{2j+1}}{(1-\lambda^2)^j} + \frac{2j}{2j+1} I - \frac{2j}{2j+1} \int_0^\lambda \frac{t^{2j}dt}{(1-t^2)^{j+1}},$$

where we used

$$\frac{t^{2j+2}}{(1-t^2)^{j+1}} = \frac{t^{2j}}{(1-t^2)^{j+1}} - \frac{t^{2j}}{(1-t^2)^j}$$

in the last equality. Thus, we get

$$I = \frac{\lambda^{2j+1}}{(1-\lambda^2)^j} - 2j \int_0^\lambda \frac{t^{2j}dt}{(1-t^2)^{j+1}}.$$

Substituting the closed formula from [15, Lemma 1] for the integral on the right-hand side of the above identity, we arrive at (5). □

For conciseness of the subsequent formulas it is convenient to introduce the parameter

$$\beta = \beta(\lambda, k) = \frac{1-\lambda^2}{1-k^2}. \quad (6)$$

Theorem 2.2. *Suppose*

$$\beta > \lambda^2 \ \Leftrightarrow\ 1 - \lambda^2 k^2 < 2(1-\lambda^2). \quad (7)$$

For each integer $N \geq 1$, we have the following decomposition

$$E(\lambda, k) = \lambda \sum_{j=0}^N (-1)^j \frac{(-1/2)_j}{j!} \left[\frac{\lambda^2}{\beta} \right]^j + \ln \left(\frac{1-\lambda}{1+\lambda} \right) \sum_{j=1}^N \frac{(-1/2)_j (1/2)_j}{j!(j-1)!} (1-k^2)^j$$
$$+ \frac{1}{\lambda} \sum_{j=1}^N \left[\frac{\lambda^2}{\beta} \right]^j \sum_{n=0}^{j-1} (-1)^{j+n-1} \frac{(-1/2)_j (1/2-j)_n}{j!(1-j)_n} \left(\frac{1-\lambda^2}{\lambda^2} \right)^n + R_{1,N}(\lambda, k). \quad (8)$$

The remainder $R_{1,N}(\lambda, k)$ satisfies the inequality

$$|R_{1,N}(\lambda, k)| \leq \frac{\lambda(1-\lambda^2)(2N-1)!!}{N 2^{N+2}(N+1)!}\left[\frac{\lambda^2}{\beta}\right]^{N+1}. \tag{9}$$

Remark 2.3. *It is apparent from the error bound* (9) *that the expansion* (8) *is convergent for any fixed λ and k satisfying* (7) *and asymptotic when $[(1-k)\lambda]/(1-\lambda) \to 0$.*

Remark 2.4. *Expansion* (8) *also converges for complex λ and k satisfying $|\lambda^2/\beta| < 1$.*

Proof. Set $k'^2 = 1 - k^2$. Expanding $\left[1 + (k'^2 t^2)/(1-t^2)\right]^{1/2}$ into the binomial series and integrating term-wise, we have

$$\begin{aligned}
E(\lambda, k) &= \int_0^\lambda \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \int_0^\lambda dt\left(1 + \frac{k'^2t^2}{1-t^2}\right)^{1/2} \\
&= \int_0^\lambda dt\left(\sum_{j=0}^{\infty}(-1)^j\frac{(-1/2)_j}{j!}\frac{k'^{2j}t^{2j}}{(1-t^2)^j}\right) \\
&= \sum_{j=0}^{N}(-1)^j\frac{(-1/2)_j}{j!}k'^{2j}\int_0^\lambda\frac{t^{2j}dt}{(1-t^2)^j} + \sum_{j=N+1}^{\infty}(-1)^j\frac{(-1/2)_j}{j!}k'^{2j}\int_0^\lambda\frac{t^{2j}dt}{(1-t^2)^j}.
\end{aligned}$$

Writing the integral in the first sum as (5), we get (8) with the remainder given by

$$\begin{aligned}
R_{1,N}(\lambda, k) &= \sum_{j=N+1}^{\infty}(-1)^j\frac{(-1/2)_j}{j!}k'^{2j}\int_0^\lambda\frac{t^{2j}dt}{(1-t^2)^j} \\
&= (-1)^{N+1}\sum_{j=0}^{\infty}(-1)^j\frac{(-1/2)_{N+j+1}}{(N+j+1)!}k'^{2(N+j+1)}\int_0^\lambda\frac{t^{2(N+j+1)}dt}{(1-t^2)^{N+j+1}} = (-1)^{N+1}\underbrace{(-a_0 + a_1 - a_2 + \cdots)}_{=S},
\end{aligned}$$

where $a_j > 0$ for all j, so that it is clearly an alternating series. The following argument shows that each term is smaller in absolute value than the previous one:

$$\begin{aligned}
\frac{(2j-1)!!}{2^{j+1}(j+1)!}(1-k^2)^{j+1}\int_0^\lambda\frac{t^{2j+2}dt}{(1-t^2)^{j+1}} &= \frac{2j-1}{2(j+1)}\frac{(2j-3)!!}{2^j j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}}{(1-t^2)^j}\frac{(1-k^2)t^2}{1-t^2}dt \\
&\leq \frac{(2j-3)!!}{2^j j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}}{(1-t^2)^j}\frac{(1-k^2)\lambda^2}{1-\lambda^2}dt \leq \frac{(2j-3)!!}{2^j j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}}{(1-t^2)^j}dt.
\end{aligned}$$

The last inequality follows from the condition (7). Thus, by the alternating series test, we see that the absolute value of the remainder $|R_{1,N}(\lambda, k)|$ is bounded by

$$a_0 = \frac{(2N-1)!!}{2^{N+1}(N+1)!}(1-k^2)^{N+1}\int_0^\lambda\frac{t^{2N+2}dt}{(1-t^2)^{N+1}}.$$

Indeed, $|R_{1,N}(\lambda, k)| = |S| = -S$ and $-a_0 \leq S \leq 0$. Next, we prove the following asymptotically exact (as $\lambda \to 1$) inequality

$$f_1(\lambda) := \int_0^\lambda\frac{t^{2b}dt}{(1-t^2)^b} \leq f_2(\lambda) := \frac{\lambda^{2b+1}}{2(b-1)(1-\lambda^2)^{b-1}}, \tag{10}$$

which is valid for each $\lambda \in (0,1)$ and $b > 1$. Indeed, $f_1(0) = f_2(0) = 0$ and

$$\frac{f_1'(\lambda)}{f_2'(\lambda)} = \frac{2(b-1)}{1+2b-3\lambda^2} < 1, \quad \lambda \in (0,1).$$

It remains to note that inequality (10) immediately implies (9). $\square$

Remark 2.5. *In [4, page 301, 903.01], Byrd and Friedman presented the following expansion*

$$E(\phi, k) = \sum_{m=0}^{\infty} \binom{1/2}{m} k'^{2m} d_{2m}(\phi), \tag{11}$$

where $\lambda = \sin(\phi)$*, and* $d_{2m}(\phi)$*'s are given by a linear recurrence relation and initial values. Since*

$$d_{2m}(\phi) = \int_0^{\sin(\phi)} \frac{t^{2m}dt}{(1-t^2)^m},$$

we see that (8) *is an equivalent form of* (11)*.*

To derive the second expansion, we will need the following lemma from [15, Lemma 2]. The symbol ${}_2F_1$ represents the Gauss hypergeometric function and P_n is Legendre's polynomial [21, section 14.7(i)].

Lemma 2.6. *(i) The function* $F_n(x) := {}_2F_1(-n, 1/2; 1; x)$ *is expressed in terms of Legendre's polynomials as:*

$$F_n(x) = (1-x)^{n/2} P_n\left(\frac{2-x}{2\sqrt{1-x}}\right). \tag{12}$$

(ii) For each $n \geq 0$*, the function* $F_n(x)$ *is decreasing on* $[0,1]$*, so that*

$$F_n(1) = \frac{(1/2)_n}{n!} \leq F_n(x) \leq F_n(0) = 1.$$

(iii) For $x \in [1,2]$*, the function* $F_n(x)$ *is monotone decreasing when* n *is an odd and satisfies the following bounds*

$$F_n(2) = 0 \leq F_n(x) \leq F_n(1) = \frac{(1/2)_n}{n!} \leq 1.$$

If n *is an even, then the function* $F_n(x)$ *has a single minimum at* $x_{\min} \in (1,2)$*, and satisfies the following bounds*

$$0 < F_n(x) \leq F_n(2) = \frac{n!}{2^n (n/2)!^2} \leq 1.$$

(iv) For $x > 2$*, the function* $F_n(x)$ *has the sign* $(-1)^n$ *and increases (decreases) for even (odd)* n*, so that*

$$|F_n(x)| \leq (x-1)^n. \tag{13}$$

We are now ready to present our second auxiliary expansion.

Theorem 2.7. *Suppose*

$$\beta k^2 < 1, \tag{14}$$

where β is defined in (6). *For each integer $N \geq 1$, we have the following decomposition*:

$$\begin{aligned} E(\lambda, k) &= E(k) - \sqrt{(1-\lambda^2)(1-k^2)} \cdot \sum_{m=0}^{N-1} \left(\frac{1}{2m+1} + \frac{\beta k^2}{2m+3} \right) (1-\lambda^2)^m \\ &\qquad \cdot {}_2F_1(-m, 1/2; 1; (1-k^2)^{-1}) + R_{2,N}(\lambda, k) \\ &= E(k) - (1-\lambda^2) \cdot \sum_{m=0}^{N-1} (-1)^m \left(\frac{k}{2m+1} + \frac{\beta k^3}{2m+3} \right) \left[\frac{k^2(1-\lambda^2)}{1-k^2} \right]^{m-1/2} \\ &\qquad \cdot {}_2F_1(-m, 1/2; 1; 1/k^2) + R_{2,N}(\lambda, k), \end{aligned} \tag{15}$$

where $E(k) = E(1, k)$ is the complete EI of the second kind. The bound for the remainder is given by

$$|R_{2,N}(\lambda, k)| \leq \frac{(N+1) \left(\beta k^2\right)^N \sqrt{(1-\lambda^2)(1-k^2)}}{(N+1/2)(N+3/2)(1-\beta k^2)} \tag{16}$$

for $1/2 \leq k^2 < 1$, and

$$|R_{2,N}(\lambda, k)| \leq \frac{(N+1)(1-\lambda^2)^N}{\lambda^2 (N+1/2)(N+3/2)} \sqrt{(1-\lambda^2)(1-k^2)} \tag{17}$$

for $0 < k^2 \leq 1/2$.

Remark 2.8. *It is clear from the error bounds* (16) *and* (17) *that the expansion* (15) *is convergent for any fixed λ and k satisfying* (14) *and is asymptotic as $(1-\lambda)/(1-k) \to 0$.*

Remark 2.9. *The set of points satisfying either condition* (7) *or condition* (14) *covers the entire unit (k, λ) square* (*see Figure 1*).

Remark 2.10. *Expansion* (15) *also holds for complex λ and k satisfying $|((1-\lambda^2)k^2)/(1-k^2)| < 1$.*

Proof. By [5, (3.1), (3.2)], we have the expansion around $\lambda = k = 0$:

$$E(\lambda, k) = \sum_{m=0}^{\infty} k^m P_n \left(\frac{k + k^{-1}}{2} \right) \lambda^{2m+1} \left[\frac{1}{2m+1} - \frac{k^2 \lambda^2}{2m+3} \right].$$

Applying item (i) of Lemma 2.6 to the above identity, we get

$$E(\lambda, k) = \sum_{m=0}^{\infty} \lambda^{2m+1} {}_2F_1(-m, 1/2; 1; 1-k^2) \left[\frac{1}{2m+1} - \frac{k^2 \lambda^2}{2m+3} \right], \tag{18}$$

which is valid for $\lambda \in (0, 1)$ and $|k| < 1/\lambda$. Next, we will employ the reflection-type relation

$$E(\lambda, k) = E(k) - \sqrt{1-k^2} \cdot E\left(\sqrt{1-\lambda^2}, \sqrt{-k^2/(1-k^2)} \right), \tag{19}$$

which can be verified by representing the second incomplete EI as the difference of the second complete EI $E(k)$ and the integral over the interval $(\lambda, 1)$ and then introducing the integration

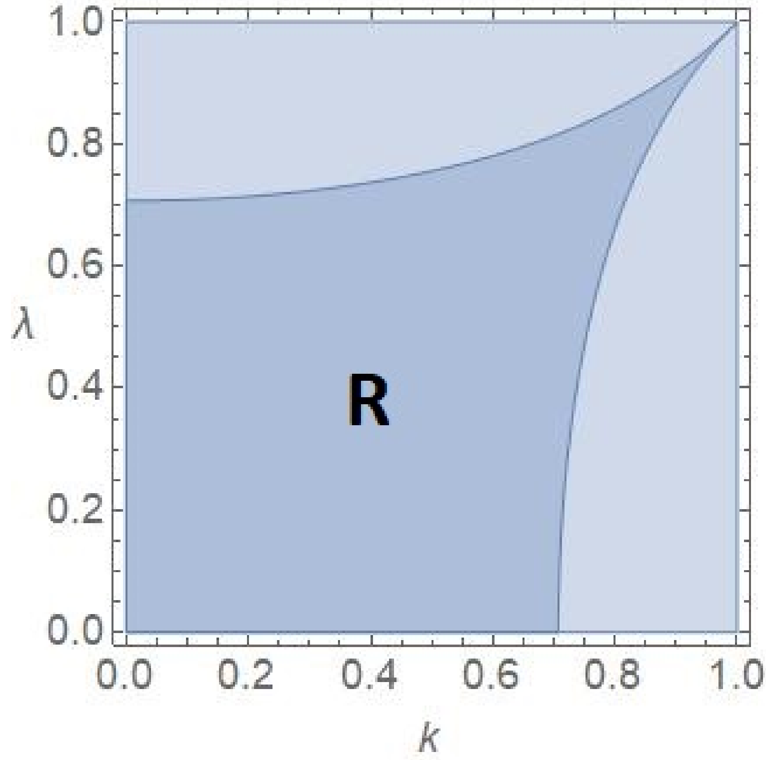

Figure 1: The set of points in the unit square satisfying (7) or (14) covers the unit square. The deep blue region R comprises the points satisfying both (7) and (14).

variable $v^2 = 1 - t^2$. Substituting (18) into the second term on the right side of (19) and splitting the corresponding series, we derive (15), which is valid for $\lambda \in (0,1)$ and $((1-\lambda^2)k^2)/(1-k^2) < 1$, with the remainder given by

$$R_{2,N}(\lambda,k) = -\sqrt{(1-\lambda^2)(1-k^2)} \sum_{m=N}^{\infty} \left(\frac{1}{2m+1} + \frac{\beta k^2}{2m+3} \right) (1-\lambda^2)^m \cdot {}_2F_1(-m, 1/2; 1; (1-k^2)^{-1}).$$

Using items (ii), (iii), (iv) of Lemma 2.6 and condition (14), we obtain

$$|R_{2,N}(\lambda,k)| \le \sqrt{(1-\lambda^2)(1-k^2)} \cdot \sum_{m=N}^{\infty} \frac{4(m+1)}{(2m+1)(2m+3)} \left(\beta k^2\right)^m$$

for $1/2 \le k^2 < 1$, and

$$|R_{2,N}(\lambda,k)| \le \sqrt{(1-\lambda^2)(1-k^2)} \sum_{m=N}^{\infty} \frac{4(m+1)}{(2m+1)(2m+3)} (1-\lambda^2)^m$$

for $0 < k^2 \leq 1/2$. Applying the following inequality

$$\begin{aligned}\sum_{m=N}^{\infty} \frac{4(m+1)x^m}{(2m+1)(2m+3)} &= x^N \sum_{s=0}^{\infty} \frac{4(N+s+1)x^s}{(2N+2s+1)(2N+2s+3)} \\ &\leq \frac{4x^N(N+1)}{(2N+1)(2N+3)} \sum_{s=0}^{\infty} x^s = \frac{4x^N(N+1)}{(2N+1)(2N+3)(1-x)},\end{aligned}$$

which is valid for $x \in (0,1)$, we get (16) and (17). Finally, the second equality in (15) follows on application of Pfaff's transformation [21, 15.8.1]. □

3 The first asymptotic expansion

For each $n \geq 0$, set

$$s_n(x) = \sum_{j=n+1}^{\infty} \frac{(-1/2)_j (1/2-j)_n}{j!(1-j)_n} (-x)^j. \tag{20}$$

Making a change of the summation variable $m = j - n - 1$, we get

$$s_n(x) = \frac{(-1/2)_{n+1}(3/2)_n}{(n+1)!n!} (-x)^{n+1} {}_4F_3 \left(\begin{matrix} 1, & 1, & 1/2+n & 3/2+n \\ & 3/2, & 1+n, & 2+n \end{matrix} \middle| -x \right), \tag{21}$$

where ${}_4F_3$ represents the generalized hypergeometric function. Note that this formula implies that $s_n(x)$ is holomorphic in the cut x-plane $\mathbb{C} \setminus (-\infty, -1]$. In particular, it is holomorphic in the unit disk $|x| < 1$ with a branch point at $x = -1$.

Next, we present a linear recurrence relation for $s_n(x)$ obtained by the method of creative telescoping [23] and the initial values in terms of elementary functions.

Lemma 3.1. *The function $s_n(x)$ satisfies the following third-order linear inhomogenous recurrence relation*

$$4(n+2)(n+3)s_{n+3}(x) = a_n(x)s_{n+2}(x) + b_n(x)s_{n+1}(x) + c_n(x)s_n(x) + d_n(x), \tag{22}$$

where

$$\begin{aligned} a_n(x) &= -(2n+3)(2nx+5x-4n-8), \\ b_n(x) &= (2n+3)(4nx+4x-2n-1), \\ c_n(x) &= -4n(1+n)x, \\ d_n(x) &= -a_n(x)g(n+2,n+3) - b_n(x)[g(n+1,n+2) + g(n+1,n+3)] \\ &\quad - c_n(x)[g(n,n+1) + g(n,n+2) + g(n,n+3)] - h(n,n+4) \end{aligned}$$

with

$$\begin{aligned} g(n,j) &= \frac{(-1/2)_j(1/2-j)_n}{j!(1-j)_n}(-x)^j, \\ h(n,n+4) &= -\frac{7}{4} \frac{(-1/2)_{n+4}(-7/2-n)_n}{(n+2)!(-3-n)_n}(-x)^{n+4} \end{aligned}$$

for each $j, n \geq 0$. The initial values are given by

$$s_0(x) = \sqrt{1+x} - 1, \tag{23}$$

$$s_1(x) = \frac{1}{4}\left[-2 + 2\sqrt{1+x} + x\left(2\ln 2 - 1 - 2\ln(1+\sqrt{1+x})\right)\right], \tag{24}$$

$$\begin{aligned} s_2(x) = &-\frac{3x^2}{16(1+\sqrt{1+x})^2(-1+\sqrt{1+x})}\Big[-\frac{1}{2}(x-\frac{8}{3})(1+\sqrt{1+x})\ln(1+\sqrt{1+x}) \\ &+\frac{1}{2}x(1+\sqrt{1+x})\ln(-1+\sqrt{1+x}) \\ &+\left((x-\frac{4}{3})\ln 2 - \frac{1}{2}x\ln x - \frac{13}{12}x + 1\right)\sqrt{1+x} + (x-\frac{4}{3})\ln 2 - \frac{1}{2}x\ln x - \frac{x}{12} - 1\Big]. \end{aligned} \tag{25}$$

Proof. Using Koutschan's `Mathematica` package `HolonomicFunctions.m` [17] that implements Chyzak's algorithm [11], we derive the following linear recurrence relation for the generic term $g(n,j)$ in (20):

$$4(n+2)(n+3)g(n+3,j) = a_n(x)g(n+2,j) + b_n(x)g(n+1,j) + c_n(x)g(n,j) + \Delta_j\left(h(n,j)\right), \tag{26}$$

where

$$\begin{aligned} h(n,j) &= -\frac{3(j-1)j(2j-2n-1)}{2(j-n-3)(j-n-2)(j-n-1)}g(n,j), \\ \Delta_j\left(h(n,j)\right) &= h(n,j+1) - h(n,j), \end{aligned}$$

for $j \geq n+4$. Taking sum in (20) with respect to j from $n+4$ to ∞, we get

$$\begin{aligned} 4(n+2)(n+3)\sum_{j=n+4}^{\infty} g(n+3,j) = a_n(x)\sum_{j=n+4}^{\infty} g(n+2,j) + b_n(x)\sum_{j=n+4}^{\infty} g(n+1,j) \\ + c_n(x)\sum_{j=n+4}^{\infty} g(n,j) + \lim_{j\to\infty} h(n,j) - h(n,n+4). \end{aligned} \tag{27}$$

Note that

$$\begin{aligned} &\sum_{j=n+4}^{\infty} g(n+3,j) = s_{n+3}(x), \\ &\sum_{j=n+4}^{\infty} g(n+2,j) = s_{n+2}(x) - g(n+2,n+3), \\ &\sum_{j=n+4}^{\infty} g(n+1,j) = s_{n+1}(x) - [g(n+1,n+2) + g(n+1,n+3)], \\ &\sum_{j=n+4}^{\infty} g(n,j) = s_n(x) - [g(n,n+1) + g(n,n+2) + g(n,n+3)], \\ &\lim_{j\to\infty} h(n,j) = 0. \end{aligned}$$

Thus, we see that (27) leads to (22).

By the `Mathematica` command "`Sum`", it is straightforward to find the closed formulae (23) and (24) for $s_0(x)$ and $s_1(x)$, respectively.

To derive a formula for $s_2(x)$ consider

$$\begin{aligned}
s_2(x) &= \sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)(3/2-j)}{j!(1-j)(2-j)}(-x)^j \\
&= \sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)}(-x)^j - \frac{1}{2}\sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)(2-j)}(-x)^j \\
&= s_1(x) + \frac{3}{16}x^2 - \frac{1}{2}\sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)(2-j)}(-x)^j \\
&= s_1(x) + \frac{3}{16}x^2 - \frac{x^2}{2}\sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)(2-j)}(-x)^{j-2}.
\end{aligned}$$

Set

$$f(x) = \sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)(2-j)}(-x)^{j-2}.$$

Then $s_2(x) = s_1(x) + \frac{3}{16}x^2 - \frac{1}{2}x^2 f(x)$. Thus, in order to derive (25), we just need a closed formula for $f(x)$. Note that

$$\begin{aligned}
(-x)^3 f'(x) &= \sum_{j=3}^{\infty} \frac{(-1/2)_j(1/2-j)}{j!(1-j)}(-x)^j \\
&= s_1(x) + \frac{3}{16}x^2.
\end{aligned}$$

Thus, we have

$$f(x) = \int_0^x \left(-\frac{1}{t^3}s_1(t) - \frac{3}{16t}\right) dt.$$

Using the `Maple` command "`int`", we obtain a closed formula for $f(x)$, which leads to (25). $\square$

Next, we present the main result of this section.

Theorem 3.2. *For each $(\lambda, k) \in (0,1) \times (0,1)$ and an integer $N \geq 1$, the second incomplete EI admits the following representation*

$$\begin{aligned}
E(\lambda, k) = \lambda\sqrt{1 + \frac{\lambda^2}{\beta}} &+ \ln\left(\frac{1-\lambda}{1+\lambda}\right) \sum_{j=1}^{N} \frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j \\
&- \frac{1}{\lambda}\sum_{n=0}^{N-1} \left(\frac{1-\lambda^2}{-\lambda^2}\right)^n s_n\left(\frac{\lambda^2}{\beta}\right) + R_N(\lambda, k), \qquad (28)
\end{aligned}$$

where β is defined in (6) and the function $s_n(x)$ is given in Lemma 3.1. Moreover, the remainder $R_N(\lambda, k)$ is negative and satisfies

$$\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!} f_{N+1}(\lambda, k) < -R_N(\lambda, k) < \frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!} f_N(\lambda, k), \qquad (29)$$

where the positive function

$$f_N(\lambda,k)=\frac{1}{1-(1-k^2)/\theta_N}\left[\frac{\theta_N}{\sqrt{\lambda^2+\beta\theta_N}}\ln\frac{\sqrt{\lambda^2+\beta\theta_N}+\lambda}{\sqrt{\lambda^2+\beta\theta_N}-\lambda}+(1-k^2)\ln\frac{1-\lambda}{1+\lambda}\right] \tag{30}$$

with

$$1<\theta_N=\frac{N(N+1)}{(N-1/2)(N+1/2)}\leq\frac{8}{3} \tag{31}$$

is strictly decreasing in N and bounded on each subset F of the unit (λ,k) square such that

$$\sup_{\lambda,k\in F}\frac{1-k}{1-\lambda}<\infty. \tag{32}$$

Remark 3.3. *The error bound (29) implies that expansion (28) is convergent at any point in the open unit square and convergence is uniform on compact subsets. Furthermore, it is asymptotic as $k\to 1$ along any curve γ lying inside the unit square with (32) satisfied, including those with the endpoint $(1,1)$.*

Remark 3.4. *Condition (32) is true for any trapezoid with vertices $(0,0)$, $(\alpha,0)$, $(1,1)$, $(0,1)$ for all $0<\alpha<1$. If condition (32) is violated but $(1-k)^n/(1-\lambda)$ remains bounded, then the n-th and higher order approximations are still asymptotic. In this case, however, it is much more efficient to employ expansion (49) from Theorem 4.2.*

Remark 3.5. *Decomposition (28) together with the inequalities (29) clearly imply a sequence of asymptotically precise two-sided bounds for the second incomplete EI in the form*

$$\begin{aligned}E_N(\lambda,k)-\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}f_N(\lambda,k)\leq E(\lambda,k)\\ \leq E_N(\lambda,k)-\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}f_{N+1}(\lambda,k),\end{aligned}$$

where $E_N(\lambda,k)=E(\lambda,k)-R_N(\lambda,k)$ in (28) and $f_N(\lambda,k)$ is defined in (30). Furthermore, we can get a substantially more precise approximation than $E_N(\lambda,k)$ by substituting $R_N(\lambda,k)$ in (28) with its approximate value read off (29). For instance, we can take

$$\hat{E}_N(\lambda,k)=E_N(\lambda,k)-\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}f_{N+\varepsilon} \tag{33}$$

with some $\varepsilon\in(0,1)$. Then it follows from (29) that the corresponding remainder $\hat{R}_N(\lambda,k):=E(\lambda,k)-\hat{E}_N(\lambda,k)$ satisfies

$$\begin{aligned}\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}\left(f_{N+\varepsilon}(\lambda,k)-f_N(\lambda,k)\right)<\hat{R}_N(\lambda,k)\\ <\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}\left(f_{N+\varepsilon}(\lambda,k)-f_{N+1}(\lambda,k)\right).\end{aligned} \tag{34}$$

We use the value $\varepsilon=1/2$ in Section 5 for numerical experiments. It seems to be an interesting open problem to find the value of $\varepsilon=\varepsilon(a)$ giving the best approximant (33) in the uniform norm on the subset of the unit square of the form $(a,1)\times(a,1)$.

Proof. By Theorem 2.2, for λ and k satisfying (7), the incomplete EI of the second kind has the following expansion

$$E(\lambda,k)=\lambda\sqrt{1+\frac{(1-k^2)\lambda^2}{1-\lambda^2}}+\ln\left(\frac{1-\lambda}{1+\lambda}\right)\sum_{j=1}^{\infty}\frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j$$
$$+\frac{1}{\lambda}\sum_{j=1}^{\infty}\sum_{n=0}^{j-1}(-1)^{j+n-1}\frac{(-1/2)_j(1/2-j)_n}{j!(1-j)_n}(1-k^2)^j\left(\frac{\lambda^2}{1-\lambda^2}\right)^{j-n}.$$

Using the rearrangement rule

$$\sum_{j=N+1}^{\infty}\sum_{n=N}^{j-1}b_{n,j}=\sum_{n=N}^{\infty}\sum_{j=n+1}^{\infty}b_{n,j} \tag{35}$$

for $N=0$, we get

$$E(\lambda,k)=\lambda\sqrt{1+\frac{(1-k^2)\lambda^2}{1-\lambda^2}}+\ln\left(\frac{1-\lambda}{1+\lambda}\right)\sum_{j=1}^{\infty}\frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j$$
$$+\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{j=n+1}^{\infty}(-1)^{j+n-1}\frac{(-1/2)_j(1/2-j)_n}{j!(1-j)_n}(1-k^2)^j\left(\frac{\lambda^2}{1-\lambda^2}\right)^{j-n}. \tag{36}$$

Taking the N-th partial sum of the first series, the $(N-1)$-th partial sum of the second series in (36), and writing the rest as a remainder, we get (28) by Lemma 3.1. Thereby, the reminder is given by

$$R_N(\lambda,k)=\ln\left(\frac{1-\lambda}{1+\lambda}\right)\sum_{j=N+1}^{\infty}\frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j$$
$$+\frac{1}{\lambda}\sum_{n=N}^{\infty}\sum_{j=n+1}^{\infty}(-1)^{j+n-1}\frac{(-1/2)_j(1/2-j)_n}{j!(1-j)_n}(1-k^2)^j\left(\frac{\lambda^2}{1-\lambda^2}\right)^{j-n}. \tag{37}$$

Applying the rule (35) to the second term in (37), we have

$$R_N(\lambda,k)=\ln\left(\frac{1-\lambda}{1+\lambda}\right)\sum_{j=N+1}^{\infty}\frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j$$
$$-\frac{1}{\lambda}\sum_{j=N+1}^{\infty}(-1)^j\frac{(-1/2)_j}{j!}\left[\frac{(1-k^2)\lambda^2}{1-\lambda^2}\right]^j\sum_{n=N}^{j-1}\frac{(1/2-j)_n}{(1-j)_n}\left(\frac{1-\lambda^2}{-\lambda^2}\right)^n. \tag{38}$$

By the argument from [15, page 197], we have that

$$\sum_{n=N}^{j-1}\frac{(1/2-j)_n}{(1-j)_n}\left(\frac{1-\lambda^2}{-\lambda^2}\right)^n=\frac{(1/2)_j}{(j-1)!}\left(\frac{1-\lambda^2}{-\lambda^2}\right)^j\left[\lambda\ln\left(\frac{1-\lambda}{1+\lambda}\right)\right.$$
$$\left.+\int_0^{\frac{\lambda^2}{1-\lambda^2}}\frac{(-u)^{j-N}du}{(1+u)\sqrt{1-u(1-\lambda^2)/\lambda^2}}\right]. \tag{39}$$

Substituting (39) into (38) and interchanging the summation and integration, we get

$$R_N(\lambda,k) = \frac{(-1)^{N+1}}{\lambda}\int_0^{\frac{\lambda^2}{1-\lambda^2}} \frac{\left(1-u(1-\lambda^2)/\lambda^2\right)^{-1/2}}{u^N(1+u)}du \sum_{j=N+1}^{\infty} \frac{(-1/2)_j(1/2)_j}{j!(j-1)!}[-(1-k^2)u]^j. \quad (40)$$

Using the `Mathematica` command "`Sum`", we find that

$$\sum_{j=N+1}^{\infty} \frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(-x)^j = \frac{(-1/2)_{N+1}(1/2)_{N+1}}{N!(N+1)!}(-x)^{N+1} \cdot {}_3F_2(1, N+1/2, N+3/2; N+1, N+2; -x). \quad (41)$$

Substituting (41) into (40), we have

$$R_N(\lambda,k) = \frac{(1-k^2)^{N+1}(-1/2)_{N+1}(1/2)_{N+1}}{\lambda N!(N+1)!} \cdot \int_0^{\frac{\lambda^2}{1-\lambda^2}} \frac{{}_3F_2\left(1, N+1/2, N+3/2; N+1, N+2; -(1-k^2)u\right)udu}{(1+u)\left(1-u(1-\lambda^2)/\lambda^2\right)^{1/2}}. \quad (42)$$

Applying the following inequality [16, Theorem 2]

$$\frac{1}{1+\frac{(N+1/2)(N+3/2)}{(N+1)(N+2)}x} < {}_3F_2(1, N+1/2, N+3/2; N+1, N+2; -x) < \frac{1}{1+\frac{(N-1/2)(N+1/2)}{N(N+1)}x} \quad \text{for each} \quad x>0$$

to (42), we see that $R_N(\lambda,k)<0$ and

$$\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}g(\theta_{N+1},\lambda,k) < -R_N(\lambda,k) < \frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!}g(\theta_N,\lambda,k),$$

where θ_N is defined in (31) and

$$\begin{aligned} g(\theta,\lambda,k) &= \frac{1-k^2}{\lambda}\int_0^{\frac{\lambda^2}{1-\lambda^2}} \frac{udu}{[1+(1-k^2)u/\theta](1+u)(1-u(1-\lambda^2)/\lambda^2)^{1/2}} \\ &= \frac{\theta}{\theta-(1-k^2)}\left[\frac{\theta}{\sqrt{\lambda^2+\beta\theta}}\cdot \ln\frac{\sqrt{\lambda^2+\beta\theta}+\lambda}{\sqrt{\lambda^2+\beta\theta}-\lambda} - (1-k^2)\ln\left(\frac{1+\lambda}{1-\lambda}\right)\right] \quad (43) \end{aligned}$$

with β from (6). Set

$$f_N(\lambda,k) = g(\theta_N,\lambda,k).$$

Then we get the error bound (29). The boundedness of $f_N(\lambda,k)$ under condition (32) can be deduced from the second representation of $g(\theta,\lambda,k)$ in (43) while the monotonicity of $f_N(\lambda,k)$ in N follows from the first representation of $g(\theta,\lambda,k)$ in (43).

We will now remove condition (7) and show that the expansion (28) is true in the entire unit square in the (λ,k) plane. Indeed, as we remarked in the introduction the function $E(\lambda,k)$ is holomorphic in the bi-disk $|\lambda|<1$, $|k|<1$ of $\mathbb{C}^2$. The same is true for the terms on the right hand side preceding $R_N(\lambda,k)$. Indeed, the hypergeometric representation (21) implies that $s_n\left(\frac{(1-k^2)\lambda^2}{1-\lambda^2}\right)$ has singularity at

$$\frac{(1-k^2)\lambda^2}{1-\lambda^2}=-1 \;\Leftrightarrow\; k^2\lambda^2=1,$$

so that s_n in (28) is also holomorphic in the bi-disk. Finally, the apparent singularity at $\lambda=0$ is removable because of $(-x)^{n+1}$ in front of ${}_4F_3$ in (21). On the other hand, the remainder $R_N(\lambda,k)$ is holomorphic in the same bi-disk due to representation (42). Hence, the difference of $E(\lambda,k)$ and the terms on the right hand side of (28) preceding $R_N(\lambda,k)$ coincide with $R_N(\lambda,k)$ under condition (7). The principle of analytic continuation implies that they coincide in the entire bi-disk. □

Remark 3.6. *By the above proof, we see that the expansion* (28) *also holds for complex* λ *and* k *in the bi-disk.*

By (28), we obtain the following first order approximation for the incomplete EI of the second kind (see details in Section 5)

$$E_1(\lambda,k)=(\lambda-1/\lambda)\sqrt{1+\lambda^2/\beta}-\frac{1-k^2}{4}\ln\left(\frac{1-\lambda}{1+\lambda}\right)+1/\lambda,$$

which is not only the correct asymptotic approximation for $E(\lambda,k)$ as $k\to 1$ but also as $\lambda\to 0$ including the case when both $\lambda,k\to 0$ along any curve inside the unit square. In fact, it is straightforward to see that

$$E_1(\lambda,k)=\lambda+\frac{1}{24}(7-10k^2+3k^4)\lambda^3+\mathcal{O}(\lambda^5),\quad \lambda\to 0.$$

On the other hand, we have

$$E(\lambda,k)=\lambda+\frac{1}{6}(1-k^2)\lambda^3+\mathcal{O}(\lambda^5),\quad \lambda\to 0.$$

Thus, we get

$$E(\lambda,k)-E_1(\lambda,k)=\mathcal{O}(\lambda^3),\quad \lambda\to 0.$$

In other words, $E_1(\lambda,k)$ is indeed an approximation for two sides of the unit square (including endpoints), *i.e.*, the side $\lambda=0,k\in[0,1]$ and the other side $k=1,\lambda\in[0,1]$. The same phenomenon happens for higher order approximations but the asymptotic order for $\lambda\to 0$ does not increase with N.

Approximation (33) with $\varepsilon=1/2$ takes the form:

$$\hat{E}_1(\lambda,k)=E_1(\lambda,k)-\frac{3(1-k^2)}{32}f_{3/2}(\lambda,k),$$

$$f_{3/2}(\lambda,k)=\frac{15}{15-8(1-k^2)}\left[\frac{15}{8\sqrt{\lambda^2+15\beta/8}}\ln\frac{\sqrt{\lambda^2+15\beta/8}+\lambda}{\sqrt{\lambda^2+15\beta/8}-\lambda}-(1-k^2)\ln\frac{1+\lambda}{1-\lambda}\right].$$

4 The second asymptotic expansion

For $n \in \mathbb{N}$ and $(\lambda, k) \in [0,1] \times [0,1)$, set

$$A_n(x) = \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(-1)^j (1/2)_j}{(2(n+j)+1)j!} x^j,$$
$$B_n(x) = \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(-1)^j (1/2)_j}{(2(n+j)+3)j!} x^j,$$
$$C_n(x) = A_n(x) + \frac{(1-\lambda^2)k^2}{1-k^2} B_n(x). \tag{44}$$

We give two representations for $A_n(x)$, $B_n(x)$ in the following lemma. The first one in terms of elementary functions serves as an ingredient of our second expansion. The second one in terms of the Gauss hypergeometric function is needed for the error estimation in Theorem 4.2 below.

Lemma 4.1. *For $n \in \mathbb{N}$ and $(\lambda, k) \in [0,1] \times [0,1)$, we have the following identities for $A_n(x)$ and $B_n(x)$:*

$$A_n(x) = \frac{1}{n!} D_x^n \left[(-1)^n \frac{(1/2)_n}{n!\sqrt{x}} \ln(\sqrt{1+x} + \sqrt{x}) + \frac{\sqrt{1+x}}{2nx} \sum_{j=0}^{n-1} (-1)^j \frac{(1/2-n)_j}{(1-n)_j} x^{n-j} \right] \tag{45a}$$
$$= \frac{1}{2x^{n+1/2}} \int_0^x \frac{t^{n-1/2}}{\sqrt{1+t}} {}_2F_1\left(-n, 1/2; 1; \frac{t}{1+t}\right) dt, \tag{45b}$$

where D_x is the usual differentiation with respect to x and the second term in the first bracket equals zero when $n = 0$; and

$$B_n(x) = \frac{1}{n!} D_x^n \left[\frac{(-1)^{n+1}(1/2)_{n+1}}{(n+1)! x^{3/2}} \ln(\sqrt{1+x} + \sqrt{x}) + \frac{\sqrt{1+x}}{2(n+1)x^2} \sum_{j=0}^{n} \frac{(-1/2-n)_j}{(-1)^j (-n)_j} x^{n+1-j} \right] \tag{45c}$$
$$= \frac{1}{2x^{n+3/2}} \int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}} {}_2F_1\left(-n, 1/2; 1; \frac{t}{1+t}\right) dt. \tag{45d}$$

Proof. The identities (45a) and (45b) for $A_n(x)$ are given in [15, Lemma 4]. Hence, we only need to derive the expressions for $B_n(x)$.

On account of $1/\sqrt{1+x} = \sum_{j=0}^{\infty} (-1)^j (1/2)_j / j! x^j$, we see that

$$B_n(x) = \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(-1)^j (1/2)_j}{(2(n+j)+3)j!} x^j = \frac{1}{2n!} D_x^n x^{-3/2} \int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}} dt. \tag{46}$$

For the integral on the right side of (46), we make a change of variables by $y^2 = t/(1+t)$ and then get

$$\int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}} dt = 2 \int_0^{\sqrt{x/(1+x)}} \frac{y^{2(n+1)} dy}{(1-y^2)^{n+2}}. \tag{47}$$

By (46), (47), and [15, Lemma 1 and 4], we derive (45c).

Alternatively, we set $t = ux$ on the right hand side of (46) and get

$$\int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}} dt = x^{n+3/2} \int_0^1 \frac{u^{n+1/2} du}{\sqrt{1+ux}}.$$

Thus, we have

$$\begin{aligned}B_n(x) &= \frac{1}{2n!}D_x^n x^{-3/2}\int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}}dt = \frac{1}{2n!}D_x^n x^n \int_0^1 \frac{u^{n+1/2}du}{\sqrt{1+ux}}\\ &= \frac{1}{2n!}\int_0^1 u^{n+1/2}\left[D_x^n \frac{x^n}{\sqrt{1+ux}}\right]du = \frac{1}{2}\int_0^1 \frac{u^{n+1/2}}{\sqrt{1+ux}}{}_2F_1\left(-n,1/2;1;\frac{ux}{1+ux}\right)du.\end{aligned}$$

Finally, substituting back $t = ux$, we obtain

$$B_n(x) = \frac{1}{2x^{n+3/2}}\int_0^x \frac{t^{n+1/2}}{\sqrt{1+t}}{}_2F_1\left(-n,1/2;1;\frac{t}{1+t}\right)dt. \tag{48}$$

Therefore, it follows from (48) and [15, Lemma 1 and 4] that (45d) holds. □

Theorem 4.2. *For each* $(\lambda,k)\in(0,1)\times(0,1)$ *and an integer* $N\geq 1$*, the second incomplete EI can be decomposed as follows:*

$$E(\lambda,k) = E(k) - \sqrt{(1-\lambda^2)(1-k^2)}\sum_{n=0}^{N-1}(1-\lambda^2)^n C_n(\beta) + \tilde{R}_N(\lambda,k), \tag{49}$$

where $\beta = (1-\lambda^2)/(1-k^2)$ *as before and the function* $C_n(x)$ *is defined in* (44) *and computed in Lemma 4.1. The remainder is negative and satisfies the following inequalities:*

$$\begin{aligned}&\frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})(1/2)_N}{2\beta^2(N+1)!}\left[\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)\right]\\ &\qquad\leq -\tilde{R}_N(\lambda,k) \leq \frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})}{2(N+1)\lambda^2\sqrt{\beta(1+\beta)}}.\end{aligned} \tag{50}$$

Remark 4.3. *The error bound* (50) *implies that expansion* (49) *is convergent for any fixed* (λ,k) *in the open unit square and convergence is uniform on compact subsets. Furthermore, it is asymptotic as* $\lambda\to 1$ *along any curve lying entirely inside the unit square, including those with the endpoint* $(1,1)$.

Remark 4.4. *Decomposition* (49) *together with the inequalities* (50) *clearly implies a sequence of asymptotically precise two-sided bounds for the second incomplete EI in the form*

$$\begin{aligned}&\tilde{E}_N(\lambda,k) - \frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})}{2(N+1)\lambda^2\sqrt{\beta(1+\beta)}} \leq E(\lambda,k)\\ &\qquad\leq \tilde{E}_N(\lambda,k) - \frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})(1/2)_N}{2\beta^2(N+1)!}\left[\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)\right],\end{aligned}$$

where $\tilde{E}_N(\lambda,k) = E(\lambda,k) - \tilde{R}_N(\lambda,k)$ *in* (49)*. Furthermore, we can get a substantially more precise approximation than* $\tilde{E}_N(\lambda,k)$ *by substituting* $\tilde{R}_N(\lambda,k)$ *in* (49) *with its approximate value. For instance, we can take the weighted average*

$$\begin{aligned}\bar{E}_N(\lambda,k) = \tilde{E}_N(\lambda,k) - \frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})}{2(N+1)}&\left(\frac{\delta}{\lambda^2\sqrt{\beta(1+\beta)}}\right.\\ &\left.+(1-\delta)\frac{(1/2)_N\left[\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)\right]}{\beta^2 N!}\right),\end{aligned} \tag{51}$$

with $0 < \delta < 1$*. Numerical experiments in section 5 give the optimal value of* $\delta = \frac{67}{187}$*. Set*

$$\bar{\Delta}_N = \frac{(1-\lambda^2)^{N+1}(\lambda^2+\beta+N^{-1})}{2(N+1)E(\lambda,k)} \left(\frac{1}{\lambda^2\sqrt{\beta(1+\beta)}} - \frac{(1/2)_N \left[\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)\right]}{\beta^2 N!} \right). \tag{52}$$

Then it follows from (50) *that the corresponding remainder* $\bar{R}_N(\lambda,k) := E(\lambda,k) - \bar{E}_N(\lambda,k)$ *satisfies*

$$-(1-\delta)\bar{\Delta}_N \cdot E(\lambda,k) \le \bar{R}_N(\lambda,k) \le \delta\bar{\Delta}_N \cdot E(\lambda,k). \tag{53}$$

Proof. By Theorem 2.7 for (λ,k) satisfying (14), we have

$$\begin{aligned} E(\lambda,k) =& E(k) - \sqrt{(1-\lambda^2)(1-k^2)} \sum_{m=0}^{\infty} (1-\lambda^2)^m \left(\frac{1}{2m+1} + \frac{\beta k^2}{2m+3}\right) {}_2F_1\left(\begin{matrix}-m, 1/2\\ 1\end{matrix} \,\middle|\, \frac{1}{1-k^2}\right) \\ =& E(k) - \sum_{m=0}^{\infty} (1-\lambda^2)^{m+1/2} \left(\frac{1}{2m+1} + \frac{\beta k^2}{2m+3}\right) \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j (1/2)_j}{j!(1-k^2)^{j-1/2}}. \end{aligned} \tag{54}$$

Applying the rule

$$\sum_{m=0}^{\infty}\sum_{j=0}^{m} b_{m,j} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_{n+j,j}$$

to (54), we get

$$\begin{aligned} E(\lambda,k) =& E(k) - \sqrt{(1-\lambda^2)(1-k^2)} \sum_{n=0}^{\infty} (1-\lambda^2)^n \\ & \times \sum_{j=0}^{\infty} \beta^j \binom{n+j}{j} \frac{(-1)^j (1/2)_j}{j!} \left(\frac{1}{2(n+j)+1} + \frac{\beta k^2}{2(n+j)+3}\right), \end{aligned} \tag{55}$$

which, in view of (44), can be split as follows:

$$E(\lambda,k) = E(k) - \sqrt{(1-\lambda^2)(1-k^2)} \sum_{n=0}^{N-1} (1-\lambda^2)^n C_n(\beta) + \tilde{R}_N(\lambda,k),$$

where the remainder is

$$\tilde{R}_N(\lambda,k) = -\sqrt{(1-\lambda^2)(1-k^2)} \sum_{n=N}^{\infty} (1-\lambda^2)^n C_n(\beta). \tag{56}$$

In (55), the inner sum does not converge unless $k < \lambda$. Nevertheless, it follows from (45) that the function $C_n(x)$ is an elementary function with no singularities in the unit square $(\lambda,k) \in [0,1]\times[0,1)$, which furnishes analytic continuation of the inner sum to this domain. Next, we prove that the outer sum of (55) converges for each (λ,k) in the unit square. For this purpose, substituting (45b) and

(45d) into (56) and applying the upper bound from item (ii) of Lemma 2.6, we get

$$
\begin{aligned}
-\tilde{R}_N(\lambda,k) &= \frac{1}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^n}{\sqrt{t(1+t)}}\,{}_2F_1\left(-n,1/2;1;\frac{t}{1+t}\right)dt\\
&\quad+\frac{k^2}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}t^n\sqrt{\frac{t}{1+t}}\,{}_2F_1\left(-n,1/2;1;\frac{t}{1+t}\right)dt\\
&\le \frac{1}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^n dt}{\sqrt{t(1+t)}}+\frac{k^2}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}t^n\sqrt{\frac{t}{1+t}}dt\\
&= \frac{1-k^2}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{dt}{\sqrt{t(1+t)}}\sum_{n=N}^{\infty}[(1-k^2)t]^n+\frac{(1-k^2)k^2}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\sqrt{\frac{t}{1+t}}dt\sum_{n=N}^{\infty}[(1-k^2)t]^n\\
&= \frac{(1-k^2)^{N+1}}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^N dt}{\sqrt{t(1+t)}[1-(1-k^2)t]}+\frac{(1-k^2)^{N+1}k^2}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{N+1/2}dt}{\sqrt{1+t}[1-(1-k^2)t]}\\
&\le \frac{(1-k^2)^{N+1}}{2\lambda^2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^N dt}{\sqrt{t(1+t)}}+\frac{(1-k^2)^{N+1}k^2}{2\lambda^2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{N+1/2}dt}{\sqrt{1+t}}\\
&= \frac{(1-k^2)^{N+1}}{\lambda^2}\int_0^{\sqrt{\frac{1-\lambda^2}{2-k^2-\lambda^2}}}\frac{y^{2N}dy}{(1-y^2)^{N+1}}+\frac{(1-k^2)^{N+1}k^2}{\lambda^2}\int_0^{\sqrt{\frac{1-\lambda^2}{2-k^2-\lambda^2}}}\frac{y^{2(N+1)}dy}{(1-y^2)^{N+2}},
\end{aligned}
\tag{57}
$$

where the last equality is derived from (47). On the other hand, it follows from [15, formula 16] that

$$
\int_0^x \frac{t^{2a}dt}{(1-t^2)^{a+1}} \le \frac{x^{2a+1}}{2a(1-x^2)^a} \quad \text{for} \quad x\in(0,1), \quad a>0. \tag{58}
$$

By setting $x=(1-\lambda^2)/(2-k^2-\lambda^2)$ and $a=N,N+1$ in (57), we obtain the upper bound in (50) by an application of (58).

To derive a lower bound, we apply the lower bound from item (ii) of Lemma 2.6 to representation

(56) with C_n expressed from Lemma 4.1 to get:

$$\begin{aligned}
-\tilde{R}_N(\lambda,k) &\geq \frac{1}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^n}{\sqrt{t(1+t)}}\frac{(1/2)_n}{n!}dt \\
&\quad+\frac{k^2}{2}\sum_{n=N}^{\infty}(1-k^2)^{n+1}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{n+1/2}}{\sqrt{1+t}}\frac{(1/2)_n}{n!}dt \\
&=\frac{1-k^2}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{dt}{\sqrt{t(1+t)}}\sum_{n=N}^{\infty}\frac{(1/2)_n}{n!}[(1-k^2)t]^n \\
&\quad+\frac{(1-k^2)k^2}{2}\int_0^{\frac{1-\lambda^2}{1-k^2}}\sqrt{\frac{t}{1+t}}dt\sum_{n=N}^{\infty}\frac{(1/2)_n}{n!}[(1-k^2)t]^n \\
&=\frac{(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^N}{\sqrt{t(1+t)}}{}_2F_1(N+1/2,1;N+1;(1-k^2)t)dt \\
&\quad+\frac{k^2(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{N+1/2}}{\sqrt{1+t}}{}_2F_1(N+1/2,1;N+1;(1-k^2)t)dt.
\end{aligned}$$

Moreover, it follows from [22, Theorem 1.10] that

$${}_2F_1(N+1/2,1;N+1;y)\geq\frac{1}{\sqrt{1-y}}\qquad\text{for}\qquad y\in(0,1).$$

Therefore, we have

$$\begin{aligned}
-\tilde{R}_N(\lambda,k) &\geq \frac{(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{N-1/2}}{\sqrt{(1+t)(1-(1-k^2)t)}}dt \\
&\quad+\frac{k^2(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}\frac{t^{N+1/2}}{\sqrt{(1+t)(1-(1-k^2)t)}}dt \\
&\geq \frac{(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}t^{N-1}\sqrt{\frac{t}{1+t}}dt \\
&\quad+\frac{k^2(1-k^2)^{N+1}(1/2)_N}{2N!}\int_0^{\frac{1-\lambda^2}{1-k^2}}t^N\sqrt{\frac{t}{1+t}}dt.
\end{aligned}$$

Using the Chebyshev inequality [20, formula IX(1.2)], we get

$$
\begin{aligned}
-\tilde{R}_N(\lambda,k) \geq{}& \frac{(1-k^2)^{N+2}(1/2)_N}{2(1-\lambda^2)N!}\int_0^{\frac{1-\lambda^2}{1-k^2}} t^{N-1}dt\int_0^{\frac{1-\lambda^2}{1-k^2}}\sqrt{\frac{t}{1+t}}dt\\
&+\frac{k^2(1-k^2)^{N+2}(1/2)_N}{2(1-\lambda^2)N!}\int_0^{\frac{1-\lambda^2}{1-k^2}} t^{N}dt\int_0^{\frac{1-\lambda^2}{1-k^2}}\sqrt{\frac{t}{1+t}}dt\\
={}&\frac{(1-k^2)^{N+2}(1/2)_N}{2(1-\lambda^2)N!}\left[\frac{1}{N}\left(\frac{1-\lambda^2}{1-k^2}\right)^N+\frac{k^2}{N+1}\left(\frac{1-\lambda^2}{1-k^2}\right)^{N+1}\right]\int_0^{\frac{1-\lambda^2}{1-k^2}}\sqrt{\frac{t}{1+t}}dt\\
={}&\frac{(1-k^2)(1-\lambda^2)^{N-1}(1/2)_N}{2N(N+1)!}\left[1-k^2+N(1-k^2\lambda^2)\right]\left[\sqrt{\frac{1-\lambda^2}{1-k^2}\left(1+\frac{1-\lambda^2}{1-k^2}\right)}-\operatorname{arcsinh}\left(\sqrt{\frac{1-\lambda^2}{1-k^2}}\right)\right]\\
={}&\frac{(1-\lambda^2)^{N+1}(1/2)_N}{2\beta^2(N+1)!}\left(\lambda^2+\beta+N^{-1}\right)\left[\sqrt{\beta\,(1+\beta)}-\operatorname{arcsinh}\left(\sqrt{\beta}\right)\right].
\end{aligned}
$$

As the expansion (49) holds for (λ, k) satisfying (14) and both sides are holomorphic for each (λ, k) in the bi-disk $|\lambda| < 1, |k| < 1$ of $\mathbb{C}^2$. It follows from the principle of analytic continuation that the expansion (49) holds in the entire bi-disk. □

Remark 4.5. *By the above proof, we see that expansion* (49) *also holds for complex λ and k in the bi-disk.*

5 Numerical experiments

In this section we will give several numerical examples of computations with the asymptotic expansions derived in Section 3 and 4. First, we consider expansion (28). By (23) and (24), we have ($\beta = (1-\lambda^2)/(1-k^2)$):

$$
\begin{aligned}
s_0\left(\frac{\lambda^2}{\beta}\right) &= \sqrt{1+\lambda^2/\beta}-1,\\
s_1\left(\frac{\lambda^2}{\beta}\right) &= \frac{1}{2}\sqrt{1+\lambda^2/\beta}-\frac{\lambda^2}{4\beta}\left[2\ln\frac{1+\sqrt{1+\lambda^2/\beta}}{2}+1\right]-\frac{1}{2}.
\end{aligned}
$$

Therefore, the first and the second order approximations are:

$$
E_1(\lambda,k) = (\lambda-1/\lambda)\sqrt{1+\lambda^2/\beta}-\frac{1-k^2}{4}\ln\left(\frac{1-\lambda}{1+\lambda}\right)+1/\lambda, \tag{59}
$$

$$
\begin{aligned}
E_2(\lambda,k) = E_1(\lambda,k) &- \frac{3}{32}(1-k^2)^2\ln\left(\frac{1-\lambda}{1+\lambda}\right)+\frac{1-\lambda^2}{2\lambda^3}\left[\sqrt{1+\lambda^2/\beta}-1\right]\\
&-\frac{1-k^2}{4\lambda}\left[2\ln\frac{1+\sqrt{1+\lambda^2/\beta}}{2}+1\right].
\end{aligned} \tag{60}
$$

The refined approximations (33) with $\varepsilon = 1/2$ take the form

$$
\hat{E}_1(\lambda,k) = E_1(\lambda,k)-\frac{3(1-k^2)}{32}f_{3/2}, \tag{61}
$$

$$\hat{E}_2(\lambda,k) = E_2(\lambda,k) - \frac{15(1-k^2)^2}{256} f_{5/2}, \tag{62}$$

where $f_N = f_N(\lambda,k)$ is defined in (30).

Denote by Δ_N the range for the relative error defined as the difference between the upper and the lower bounds in (29) divided by $E(\lambda,k)$:

$$\Delta_N = \frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!E(\lambda,k)}(f_N(\lambda,k) - f_{N+1}(\lambda,k)). \tag{63}$$

Approximation (61) together with (34) places the value of $E(\lambda,k)$ within an interval of length $\Delta_1 \cdot E(\lambda,k)$, while (62) places $E(\lambda,k)$ within an interval of length $\Delta_2 \cdot E(\lambda,k)$. The results of numerical computations are presented in Table 1. The exact values of $E(\lambda,k)$ shown in the tables below are computed using `Mathematica` with the required number of precise digits guaranteed.

λ	k	$E(\lambda,k)$	First order approx. (59)	First order approx. (61)	Relative error e_1	Relative error $\hat{e}_1$	Relative error range Δ_1
.8	.8	.8501	.8714	.8496	$-.02504$	$.6011 \times 10^{-3}$	.002446
.9	.9	.9504	.9669	.9500	$-.01734$	$.4455 \times 10^{-3}$	.001972
.95	.95	.9900	1.0003	.9897	$-.01044$	$.2712 \times 10^{-3}$	.001250
.99	.99	1.0056	1.0081	1.0055	$-.002531$	$.6475 \times 10^{-4}$	$.3072 \times 10^{-3}$
.95	.99	.9586	.9591	.9586	$-.5674 \times 10^{-3}$	$.5651 \times 10^{-7}$	$.1743 \times 10^{-4}$
.99	.999	.9916	.9916	.9916	$-.3417 \times 10^{-4}$	$.1902 \times 10^{-8}$	$.5445 \times 10^{-8}$
λ	k	$E(\lambda,k)$	Second order approx. (60)	Second order approx. (62)	Relative error e_2	Relative error $\hat{e}_2$	Relative error range Δ_2
.8	.8	.8501	.8547	.8501	$-.005413$	$.4975 \times 10^{-4}$	$.1990 \times 10^{-3}$
.9	.9	.9504	.9523	.9504	$-.001966$	$.1837 \times 10^{-4}$	$.8270 \times 10^{-4}$
.95	.95	.9900	.9906	.9900	$-.6056 \times 10^{-3}$	$.5978 \times 10^{-7}$	$.2661 \times 10^{-4}$
.99	.99	1.0056	1.0056	1.0056	$-.2995 \times 10^{-4}$	$.2667 \times 10^{-8}$	$.1324 \times 10^{-7}$
.95	.99	.9586	.9586	.9586	$-.6968 \times 10^{-7}$	$.3090 \times 10^{-10}$	$.8609 \times 10^{-9}$
.99	.999	.9916	.9916	.9916	$-.4240 \times 10^{-9}$	$.1081 \times 10^{-11}$	$.2810 \times 10^{-11}$

Table 1: Numerical examples for approximations (59), (61), (60) and (62) derived from expansion (33). The numbers e_i and $\hat{e}_i$ are the relative errors $(E(\lambda,k) - E_i(\lambda,k))/E(\lambda,k)$ and $(E(\lambda,k) - \hat{E}_i(\lambda,k))/E(\lambda,k)$, respectively, $i = 1,2$. The numbers Δ_1, Δ_2 are given in (63).

Next, we consider expansion (49). By (45a) and (45c) we have

$$\begin{aligned} C_0(x) &= \frac{1}{\sqrt{x}}\ln(\sqrt{1+x}+\sqrt{x}) + \frac{\beta k^2}{2x^{3/2}}\left(\sqrt{x(1+x)} - \ln(\sqrt{1+x}+\sqrt{x})\right), \\ C_1(x) &= \frac{1}{4x}\left(\frac{1}{\sqrt{x}}\ln(\sqrt{1+x}+\sqrt{x}) - \frac{1-x}{\sqrt{1+x}}\right) \\ &\quad + \frac{\beta k^2}{16x^2}\left(\frac{-9}{\sqrt{x}}\ln(\sqrt{1+x}+\sqrt{x}) + \frac{9+3x+2x^2}{\sqrt{1+x}}\right). \end{aligned} \tag{64}$$

Hence, the first and the second order approximation obtained from (49) are:

$$\tilde{E}_1(\lambda,k) = E(k) - \frac{2-3k^2+k^4}{2}\ln\left(\sqrt{1+\beta}+\sqrt{\beta}\right) - \frac{k^2(1-k^2)}{2}\sqrt{\beta(1+\beta)}, \tag{65}$$

$$\tilde{E}_2(\lambda,k) = \tilde{E}_1(\lambda,k) - (1-\lambda^2)^{3/2}\sqrt{1+k}\cdot C_1(\beta), \tag{66}$$

where the function $C_1(x)$ is given in (64). The refined approximations (51) take the form

$$\bar{E}_1(\lambda,k) = \tilde{E}_1(\lambda,k) - \frac{(1-\lambda^2)^2(\lambda^2+\beta+1)}{4}\left(\frac{67}{187}\frac{1}{\lambda^2\sqrt{\beta(1+\beta)}} + \frac{60}{187}\frac{\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)}{\beta^2}\right), \tag{67}$$

$$\bar{E}_2(\lambda,k) = \tilde{E}_2(\lambda,k) - \frac{(1-\lambda^2)^3(\lambda^2+\beta+1/2)}{6}\left(\frac{67}{187}\frac{1}{\lambda^2\sqrt{\beta(1+\beta)}} + \frac{45}{187}\frac{\sqrt{\beta(1+\beta)} - \operatorname{arcsinh}\left(\sqrt{\beta}\right)}{\beta^2}\right). \tag{68}$$

Let $\bar{\Delta}_N$ be the number given in (52). Then approximation (67) together with (53) puts $E(\lambda,k)$ within an interval of length $\bar{\Delta}_1 \cdot E(\lambda,k)$, while (68) puts $E(\lambda,k)$ within an interval of length $\bar{\Delta}_2 \cdot E(\lambda,k)$. The results of numerical computation are presented in Table 2.

λ	k	$E(\lambda,k)$	First order approx. (65)	First order approx. (67)	Relative error $\tilde{e}_1$	Relative error $\bar{e}_1$	Relative error range $\bar{\Delta}_1$
.8	.8	.8501	.8976	.8491	$-.05586$	.001162	.08435
.9	.9	.9504	.9532	.9509	$-.01343$	$-.5311\times10^{-3}$	.01618
.95	.95	.9900	.9933	.9909	$-.003344$	$-.9083\times10^{-3}$	.003602
.99	.95	1.0572	1.0574	1.0572	$-.2771\times10^{-3}$	$-.3481\times10^{-4}$	$.2784\times10^{-3}$
.99	.99	1.0056	1.0057	1.0056	$-.1355\times10^{-3}$	$-.3648\times10^{-4}$	$.1335\times10^{-3}$
.999	.99	1.0220	1.0220	1.0220	$-.4114\times10^{-7}$	$-.5547\times10^{-8}$	$.4085\times10^{-7}$
λ	k	$E(\lambda,k)$	Second order approx. (66)	Second order approx. (68)	Relative error $\tilde{e}_2$	Relative error $\bar{e}_2$	Relative error range $\bar{\Delta}_2$
.8	.8	.8501	.8589	.8501	$-.01028$	$-.2378\times10^{-4}$	.01771
.9	.9	.9504	.9516	.9505	$-.001286$	$-.6168\times10^{-4}$	.001870
.95	.95	.9900	.9901	.9900	$-.1633\times10^{-3}$	$-.1004\times10^{-4}$	$.2188\times10^{-3}$
.99	.95	1.0572	1.0572	1.0572	$-.3110\times10^{-7}$	$-.8657\times10^{-8}$	$.3212\times10^{-7}$
.99	.99	1.0056	1.0056	1.0056	$-.1345\times10^{-7}$	$-.9252\times10^{-9}$	$.1689\times10^{-7}$
.999	.99	1.0220	1.0220	1.0220	$-.4774\times10^{-10}$	$-.1502\times10^{-10}$	$.4679\times10^{-10}$

Table 2: Numerical examples for approximations (65), (67), (66) and (68) derived from expansion (51). The numbers $\tilde{e}_i$ and $\bar{e}_i$ are the relative errors $(E(\lambda,k)-\tilde{E}_i(\lambda,k))/E(\lambda,k)$ and $(E(\lambda,k)-\bar{E}_i(\lambda,k))/E(\lambda,k)$, respectively, $i=1,2$. The numbers $\bar{\Delta}_1, \bar{\Delta}_2$ are given in (52).

We conclude by comparing the above results with the asymptotic approximation (4) from [18, 10] with the error bounds (72) and (73), respectively. We denote Δ_1^* to be the difference between the upper and the lower bound in (72) divided by $E(\lambda,k)$, *i.e.*,

$$\Delta_1^* = (\text{rhs of } (72)\ -\ \text{lhr of } (72))/E(\lambda,k). \tag{69}$$

Similarly, Let Δ_2^* be the difference between the upper and the lower bound in (73) divided by $E(\lambda,k)$, *i.e.*,

$$\Delta_2^* = (\text{rhs of } (73)\ -\ \text{lhr of } (73))/E(\lambda,k). \tag{70}$$

The results of numerical computation are given in Table 3.

λ	k	$E(\lambda, k)$	First order approx. (4)	Relative error	Relative error range Δ_1^*	Relative error range Δ_2^*
.8	.8	.8501	.8343	.01864	.78055	.23538
.9	.9	.9504	1.0127	$-.06551$	.66727	.17444
.95	.95	.9900	1.0704	$-.08121$	.44780	.12025
.99	.95	1.0572	1.1178	$-.05736$	.27546	.105
.99	.99	1.0056	1.0472	$-.04136$	.15715	.03994
.999	.99	1.0220	1.0434	$-.02088$	.07386	.03581
.999	.999	1.0017	1.0094	$-.007712$	.02327	.006137

Table 3: Numerical examples for the approximation (4) (=(71)) due to Carlson-Gustafson and López. The fifth column equals the relative error r_1 in (4) (=(71)) divided by $E(\lambda, k)$. The numbers Δ_1^* and Δ_2^* are given in (69) and (70), respectively.

Acknowledgements We thank the anonymous referees for a number of useful remarks which improve the exposition of the paper.

References

[1] Bille C. Carlson's Profile. https://dlmf.nist.gov/about/bio/BCCarlson.

[2] Digital Library of Mathematical Functions, Chapter 19, Elliptic Integrals. https://dlmf.nist.gov/19.

[3] S. I. Bezrodnykh. Analytic Continuation of the Appell Function F_1 and Integration of the Associated System of Equations in the Logarithmic Case. *Computational Mathematics and Mathematical Physics*, 57 (4):559–589, 2017.

[4] P. F. Byrd and M. D. Friedman. *Handbook of Elliptic Integrals for Engineers and Scientists.* Springer, Berlin, Heidelberg, 1971.

[5] B. C. Carlson. Some series and bounds for incomplete elliptic integrals. *Journal of Mathematics and Physics*, 40:125–134, 1961.

[6] B. C. Carlson. *Special Functions of Applied Mathematics.* Academic Press, New York, 1977.

[7] B. C. Carlson. Computing elliptic integrals by duplication. *Numerische Mathematik*, 33:1–16, 1979.

[8] B. C. Carlson. The hypergeometric function and the R-function near their branch points. *Rend. Sem. Mat. Univ. Politec. Torino*, special issue:63–89, 1985.

[9] B. C. Carlson and J. L. Gustafson. Asymptotic expansion of the first elliptic integral. *SIAM Journal on Mathematical Analysis*, 16 (5):1072–1092, 1985.

[10] B. C. Carlson and J. L. Gustafson. Asymptotic approximations for symmetric elliptic integrals. *SIAM Journal on Mathematical Analysis*, 25 (2):288–303, 1994.

[11] F. Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Math.*, 217(1-3):115–134, 2000. Formal power series and algebraic combinatorics (Vienna, 1997).

[12] H. V. de Vel. On the series expansion method for computing incomplete elliptic integrals of the first and second kinds. *Mathematics of Computation*, 23 (105):61–69, 1969.

[13] E. L. Kaplan. Auxiliary table for the incomplete elliptic integrals. *Journal of Mathematics and Physics*, 27:11–36, 1948.

[14] D. Karp. An approximation for zero-balanced Appell function F_1 near $(1, 1)$. *Journal of Mathematical Analysis and Applications*, 339:1332–1341, 2008.

[15] D. Karp and S. M. Sitnik. Asymptotic approximations for the first incomplete elliptic integral near logarithmic singularity. *Journal of Computational and Applied Mathematics*, 205 (1):186–206, 2007.

[16] D. Karp and S. M. Sitnik. Inequalities and monotonicity of ratios for generalized hypergeometric function. *Journal of Approximation Theory*, 161 (1):337–352, 2009.

[17] C. Koutschan. HolonomicFunctions user's guide. *RISC Report Series*, pages 1–93, 2010.

[18] J. L. López. Asymptotic expansions of symmetric standard elliptic integrals. *SIAM Journal on Mathematical Analysis*, 31 (4):754–775, 2000.

[19] J. L. López. Uniform asymptotic expansions of symmetric elliptic integrals. *Constructive Approximation*, 17 (4):535–559, 2001.

[20] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink. *Classical and New Inequalities in Analysis.* Kluwer Academic Publishers, Dordrecht, 1993.

[21] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, New York, 2010.

[22] S. Ponnusamy and M. Vuorinen. Asymptotic expansions and inequalities for hypergeometric function. *Mathematica*, 44 (2):278–301, 1997.

[23] D. Zeilberger. The method of creative telescoping. *Journal of Symbolic Computation*, 11(3):195–204, 1991.

Appendix. Approximations of Carlson-Gustafson and López

In this appendix we will convert the asymptotic approximations for symmetric elliptic integrals R_F and R_D due to López [18] and Carlson-Gustafson [10] and their error bounds into the corresponding results for the Legendre's second incomplete EI $E(\lambda, k)$ defined in (1). As

$$E(\lambda, k) = \lambda R_F(1-\lambda^2, 1-k^2\lambda^2, 1) - \frac{1}{3}k^2\lambda^3 R_D(1-\lambda^2, 1-k^2\lambda^2, 1)$$

by [7, (4.2)], we need the asymptotic approximations for $R_F(a, b, 1)$ and $R_D(a, b, 1)$ as $a, b \to 0$. In view of the easily verifiable relations

$$R_F(x, y, z) = z^{-1/2} R_F(x/z, y/z, 1), \quad R_D(x, y, z) = z^{-3/2} R_D(x/z, y/z, 1),$$

it suffices to use the asymptotics of $R_F(x,y,z)$ and $R_D(x,y,z)$ as $z\to\infty$ while x and y remain fixed. The first approximations from [18, (3.1)] after simple rearrangement are given by (under the assumption $0\le x<y\le z$)

$$R_F(x/z,y/z,1)=\ln\left(\frac{2}{\sqrt{x/z}+\sqrt{y/z}}\right)+(\psi(1)-\psi(1/2))/2+R_1^F$$

with the error bound [18, (3.5)] (we used the identity $\psi(2)=\psi(1)+1$):

$$0<R_1^F\le\frac{|x+y|}{8z}\left(\ln\left(1+\frac{2z}{|x+y|}\right)+2\right),$$

$$0<R_1^F\le\frac{\max(2,|x+y|)}{\sqrt{z}}.$$

Further, by [18, (3.14)] (also under the assumption $0\le x<y\le z$)

$$R_D(x/z,y/z,1)=3\ln\left(\frac{2}{\sqrt{x/z}+\sqrt{y/z}}\right)+\frac{3}{2}\left(\psi(1)-\psi(3/2)\right)+R_1^D$$

with

$$0<R_1^D\le\frac{9|x+y|}{8z}\left(\ln\left(1+\frac{2z}{|x+y|}\right)+2\right),$$

$$0<R_1^D\le\frac{2\max(2,|x+y|)}{\sqrt{z}}.$$

Hence, if we set $1-k^2\lambda^2=x/z$ and $1-\lambda^2=y/z$ we ensure that $0\le x<y$ and we arrive at (in view of $\psi(3/2)=\psi(1/2)+2$):

$$E(\lambda,k)=\lambda(1-k^2\lambda^2)\left[\ln\left(\frac{2}{\sqrt{1-k^2\lambda^2}+\sqrt{1-\lambda^2}}\right)+\frac{1}{2}(\psi(1)-\psi(1/2))\right]+k^2\lambda^3+r_1.$$

Finally, due to $\sum_{r=1}^{m}\psi(r/m)=m\psi(1)-m\ln(m)$ for $m=2$ we have $\psi(1)-\psi(1/2)=\ln(4)$ and the approximation takes the form

$$E(\lambda,k)=\lambda(1-k^2\lambda^2)\ln\left(\frac{4}{\sqrt{1-k^2\lambda^2}+\sqrt{1-\lambda^2}}\right)+k^2\lambda^3+r_1 \tag{71}$$

reproduced in (4) with the error bounds

$$-\frac{3k^2\lambda^3(2-\lambda^2-k^2\lambda^2)}{8}\left[\ln\frac{4-\lambda^2-k^2\lambda^2}{2-\lambda^2-k^2\lambda^2}+2\right]\le r_1\le\frac{\lambda(2-\lambda^2-k^2\lambda^2)}{8}\left[\ln\frac{4-\lambda^2-k^2\lambda^2}{2-\lambda^2-k^2\lambda^2}+2\right]. \tag{72}$$

Combining [10, (26)] and [10, (34)] we again obtain the approximation (71). However, the bounds for remainder term differ from those above and take the form

$$\left(\frac{\lambda\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}{2\left(1-\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}\right)}-\frac{3k^2\lambda^3(2-\lambda^2(1+k^2))}{2\lambda^2(1+k^2)}\right)\ln\frac{2}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}}<r_1$$

$$<\frac{\lambda(2-\lambda^2(1+k^2))}{2+\lambda^2(1+k^2)}\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}}-\frac{k^2\lambda^3\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}{1-\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}\ln\frac{2}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}}. \tag{73}$$

Table 3 shows that these error bounds are more precise than (72) at the price of being substantially more complicated.