Rational Solutions of First-Order Algebraic Ordinary Difference Equations

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AOΔE

Let $\mathbb{K}$ be an algebraic closed field of char 0, and x be an indeterminate.

Consider the algebraic ordinary difference equation (AOΔE):

$$F(x, y(x), y(x+1), \cdots, y(x+m)) = 0, \tag{1}$$

where F is a polynomial in $y(x), y(x+1), \ldots, y(x+m)$ with coeffs in $\mathbb{K}(x)$ and $m \in \mathbb{N}$ is called the order of F. We also simply write (1) as $F(y) = 0$. An AOΔE is autonomous if x does not appear in it explicitly.

Example 1. Equations of Riccati type:

$$y(x+1)y(x) + p(x)y(x+1) + q(x)y(x) = 0,$$

where $p, q \in \mathbb{K}[x]$.

Motivation

Goal: Given a first-order AOΔE $F(y) = 0$. Determine a strong rational general solution $s \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$, where c is transcendental over $\mathbb{K}(x)$, s.t.

$$F(x, s(x), s(x+1)) = 0.$$

Let $s(x) = \frac{p(x)}{q(x)}$ with $\gcd(p, q) = 1$. Denote the degree of s by $\deg(s) := \max(\deg(p), \deg(q))$.

Applications:

- Automatic proof of combinatorial identities: symbolic summation.
- Difference Galois theory: factorization of linear difference operators.
- Analysis of time or space complexity of computer programs.

Motivation

Previous works:

- (Abramov-Bronstein-Petkovšek-van Hoeij 1989-1998): Algorithms for computing rational solutions of linear difference equations.
- (Feng-Gao-Huang 2008): An algorithm for computing rational solutions of first-order autonomous AOΔEs provided the degree of the rational solution is given.
- (Shkaravska-Eekelen 2014, 2021): a degree bound for polynomial solutions of high-order non-autonomous AOΔEs under a sufficient condition.

Our contribution: Construct a degree bound for rational solutions of first-order autonomous AOΔEs, thus derive a complete algorithm for computing corresponding rational solutions.

Preliminaries

Let $F \in \mathbb{K}[x, y, z] \setminus \{0\}$ be an irreducible polynomial.

Recall: A solution s of the AOΔE $F(x, y(x), y(x+1)) = 0$ is called a strong rational general solution if $s \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$ for some c which is transcendental over $\mathbb{K}(x)$.

Theorem 1: If the AOΔE $F(x, y(x), y(x+1)) = 0$ admits a strong rational general solution, then the algebraic curve in $\mathbb{A}^2\left(\overline{\mathbb{K}(x)}\right)$ defined by $F(x, y, z) = 0$ is of genus zero.

Definition 1: The algebraic curve $\mathcal{C}_F \subset \mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F(x, y, z) = 0$ is called the corresponding algebraic curve of the AOΔE $F(x, y(x), y(x+1)) = 0$.

Preliminaries

Using parametrization theory of rational curves, we have

Proposition 1: If the algebraic curve $\mathcal{C}_F \subset \mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F(x, y, z) = 0$ is of genus zero, then there exists a birational transformation $\mathcal{P} : \mathbb{A}^1(\overline{\mathbb{K}(x)}) \to \mathcal{C}_F$ defined by $\mathcal{P}(x, t) = (p_1(x, t), p_2(x, t))$ for some $p_1(x, t), p_2(x, t) \in \mathbb{K}(x, t)$.

- There exists an algorithm (Vo-Grasegger-Winkler 2018) for determining such a birational transformation as above.

Preliminaries

Theorem 2: Let $F(x, y(x), y(x+1)) = 0$ be an AOΔE s.t. its corresponding curve $\mathcal{C}_F$ is of genus zero. Assume $\mathcal{P}(x,t) = (p_1(x,t), p_2(x,t)) \in \mathbb{K}(x,t)^2$ is a birational transformation from $\mathbb{A}^1(\overline{\mathbb{K}(x)})$ to $\mathcal{C}_F$. Consider

$$p_1(x+1, \omega(x+1)) = p_2(x, \omega(x)). \tag{2}$$

- If $s(x, c)$ is a strong rational general solution of $F(y) = 0$, then there exists a strong rational general solution $\omega(x, c)$ of (2) s.t. $s(x, c) = p_1(x, \omega(x, c))$.
- Conversely, if $\omega(x, c)$ is a strong rational general solution of (2), then $s(x, c) = p_1(x, \omega(x, c))$ is a strong rational general solution of $F(y) = 0$.

We call (2) an associated separable AOΔE of $F(y) = 0$.

Preliminaries

Proposition 2: If the AOΔE $F(x, y(x), y(x+1)) = 0$ admits a strong rational general solution, then we have

$$\deg_y F = \deg_z F.$$

In this case, the associated separable AOΔE exists and it must be of the form

$$P(x, \omega(x+1)) = Q(x, \omega(x)),$$

for some $P, Q \in \mathbb{K}(x, y)$ s.t.

$$\deg_y P = \deg_y Q = \deg_z F = \deg_y F.$$

Goal: Construct a degree bound for rational solutions of autonomous separable AOΔEs, and thus derive an algorithm for computing rational solutions of first-order aotonomous AOΔEs.

Difference Riccati equations

Consider the first-degree autonomous separable AOΔE:

$$\frac{a_1 y(x+1) + b_1}{c_1 y(x+1) + d_1} = \frac{a_2 y(x) + b_2}{c_2 y(x) + d_2}, \tag{3}$$

where

1. $a_1 d_1 - c_1 b_1 \neq 0$ and $a_2 d_2 - c_2 b_2 \neq 0$;
2. $a_1 \neq 0$ or $c_1 \neq 0$;
3. $a_2 \neq 0$ or $c_2 \neq 0$.

We call (3) a difference Riccati equation, which can be transformed into a second-oder linear OΔE. We present another way to compute its rational solutions, which can be generalized to arbitrary degree separable AOΔEs.

Difference Riccati equations

Let $\frac{A(x)}{B(x)} \in \mathbb{K}(x)$ be a solution of (3) with $\gcd(A(x), B(x)) = 1$. Substituting $\frac{A(x)}{B(x)}$ into (3) , we get

$$\frac{a_1 A(x+1) + b_1 B(x+1)}{c_1 A(x+1) + d_1 B(x+1)} = \frac{a_2 A(x) + b_2 B(x)}{c_2 A(x) + d_2 B(x)}. \tag{4}$$

By a gcd argument, we see that (4) is equivalent to

$$\begin{cases} a_1 A(x+1) + b_1 B(x+1) = c \cdot (a_2 A(x) + b_2 B(x)), \\ c_1 A(x+1) + d_1 B(x+1) = c \cdot (c_2 A(x) + d_2 B(x)) \end{cases} \tag{5}$$

for some unknown $c \in \mathbb{K} \setminus \{0\}$.

By doing coefficient comparison, we can determine finite candidates for c algorithmically. WLOG, we assume that $c = 1$.

Difference Riccati equations

Consider

$$a_1 A(x+1) + b_1 B(x+1) = a_2 A(x) + b_2 B(x), \tag{6}$$
$$c_1 A(x+1) + d_1 B(x+1) = c_2 A(x) + d_2 B(x). \tag{7}$$

Taking $c_1 \times (6) - a_1 \times (7)$, we get

$$(a_1 d_1 - b_1 c_1) B(x+1) = (a_1 c_2 - a_2 c_1) A(x) + (a_1 d_2 - b_2 c_1) B(x). \tag{8}$$

Taking $c_2 \times (6) - a_2 \times (7)$ and applying $\sigma^{-1} : x \longmapsto x - 1$ to it, we have

$$(a_2 d_2 - b_2 c_2) B(x-1) = (a_2 c_1 - a_1 c_2) A(x) + (a_2 d_1 - b_1 c_2) B(x). \tag{9}$$

Difference Riccati equations

Taking $(8)+(9)$, we see that $B(x)$ is a polynomial solution of the second-order linear OΔE:

$$(a_1 d_1 - b_1 c_1) f(x+2) + (b_2 c_1 + b_1 c_2 - a_2 d_1 - a_1 d_2) f(x+1) \\ + (a_2 d_2 - c_2 b_2) f(x) = 0, \quad (10)$$

where $f(x)$ is unknown and $a_i d_i - b_i c_i \neq 0$ for $i \in \{1, 2\}$.

Similarly, we can show that $A(x)$ also satisfies (10).

Assume $\{p_0(x), p_1(x)\}$ is a $\mathbb{K}$-basis of polynomial solutions of (10), which is implemented in `Maple`.

Difference Riccati equations

Then it follows from (10) that

$$A(x) = \ell_0 p_0(x) + \ell_1 p_1(x) \quad \text{and} \quad B(x) = \ell_2 p_0(x) + \ell_3 p_1(x), \tag{11}$$

where $\ell_i \in \mathbb{K}$ is to be determined, $i = 0, \ldots, 3$.

Substituting (11) into

$$\begin{aligned} a_1 A(x+1) + b_1 B(x+1) &= a_2 A(x) + b_2 B(x), \\ c_1 A(x+1) + d_1 B(x+1) &= c_2 A(x) + d_2 B(x). \end{aligned}$$

and solving the corresponding linear equations for ℓ_i's, we find rational solutions of difference Riccati equations.

Problem

Question 1: Let P_1, P_2, Q_1, Q_2 be polynomials in $\mathbb{K}[z] \setminus \{0\}$ such that $\gcd(P_1, Q_1) = \gcd(P_2, Q_2) = 1$ and $\deg \frac{P_1}{Q_1} = \deg \frac{P_2}{Q_2} = n \geq 1$. Find all rational solutions of the autonomous separable AOΔE

$$\frac{P_1(y(x+1))}{Q_1(y(x+1))} = \frac{P_2(y(x))}{Q_2(y(x))}. \tag{12}$$

If $n = 1$, then (12) is the difference Riccati equation.

Reduction

By a gcd argument, we have

Proposition 3: Let P_1, P_2, Q_1, Q_2 be polynomials specified in Problem 1. Set

$$\tilde{P}_i(z, w) = w^n P_i\left(\frac{z}{w}\right), \quad \text{and} \quad \tilde{Q}_i(z, w) = w^n Q_i\left(\frac{z}{w}\right),$$

which are homogeneous of degree n in $\mathbb{K}[z, w]$, $i = 1, 2$. Assume $\frac{A(x)}{B(x)}$ is a solution of (12), where $A, B \in \mathbb{K}[x]$ with $\gcd(A, B) = 1$. Then there exists $c \in \mathbb{K}$ s.t.

$$\begin{cases} \tilde{P}_1(A(x+1), B(x+1)) = c \cdot \tilde{P}_2(A(x), B(x)), \\ \tilde{Q}_1(A(x+1), B(x+1)) = c \cdot \tilde{Q}_2(A(x), B(x)). \end{cases} \tag{13}$$

By doing coefficient comparison, we can determine finite candidates for c algorithmically. WLOG, we assume that $c = 1$.

Uncoupling

$$
\begin{cases} \tilde{P}_1(A(x+1), B(x+1)) = \tilde{P}_2(A(x), B(x)), \\ \tilde{Q}_1(A(x+1), B(x+1)) = \tilde{Q}_2(A(x), B(x)). \end{cases} \tag{14}
$$

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Applying $\sigma : x \longmapsto x+1$ to the above equations, we get

$$\begin{cases} \tilde{P}_1(A(x+2), B(x+2)) = \tilde{P}_2(A(x+1), B(x+1)), \\ \tilde{Q}_1(A(x+2), B(x+2)) = \tilde{Q}_2(A(x+1), B(x+1)). \end{cases} \tag{15}$$

Regarding $A(x+i)$ and $B(x+i)$ as undeterminates, we have 4 equations and 6 variables. It is possible to utilize nonlinear elimination techniques to eliminate 3 variables, *i.e.*, $A(x+i)$'s or $B(x+i)$'s from (14) and (15).

Uncoupling

Algorithm 1: Given the difference system (13). Compute nonzero autonomous second-order AOΔEs for $A(x)$ and $B(x)$, respectively, which are consequences of (13).

(1) Let $I \subseteq \mathbb{K}[w_0, w_1, w_2, z_0, z_1, z_2]$ be the ideal generated by

$$\tilde{P}_1(z_1, w_1) - \tilde{P}_2(z_0, w_0), \quad \tilde{Q}_1(z_1, w_1) - \tilde{Q}_2(z_0, w_0),$$
$$\tilde{P}_1(z_2, w_2) - \tilde{P}_2(z_1, w_1), \quad \tilde{Q}_1(z_2, w_2) - \tilde{Q}_2(z_1, w_1).$$

Using Gröbner bases or resultants, compute nonzero elements $F_A \in I \cap \mathbb{K}[z_0, z_1, z_2]$ and $F_B \in I \cap \mathbb{K}[w_0, w_1, w_2]$.

(2) Return $F_A(A(x), A(x+1), A(x+2)) = 0$ and $F_B(B(x), B(x+1), B(x+2)) = 0$.

Uncoupling

Theorem 3 (Vo-Z. 2020) The elimination ideals $I \cap \mathbb{K}[z_0, z_1, z_2]$ and $I \cap \mathbb{K}[w_0, w_1, w_2]$ are nonzero and Algorithm 1 is correct.

Ingredients for the proof:

- Properties of resultants.
- weak version of Hilbert Nullstellensatz.

Polynomial solutions

Let $\frac{A(x)}{B(x)}$ be a solution of the autonomous separable OΔE. By Algorithm 1, we can find nonzero autonomous second-order AOΔEs for $A(x)$ and $B(x)$, respectively.

Question 2: Let $F \in \mathbb{K}[y, z, w]$ be a homogeneous polynomial. Find all polynomial solutions of the AOΔE

$$F(y(x), y(x+1), y(x+2)) = 0. \tag{16}$$

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Idea: Doing coefficient comparison to derive a degree bound.

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$$F(y(x), y(x+1), y(x+2)) = 0. \tag{16}$$

Idea: Doing coefficient comparison to derive a degree bound. Note that (16) is equivalent to

$$\tilde{F}(y(x), \Delta y(x), \Delta^2 y(x)) = 0, \tag{17}$$

where $\Delta y(x) = y(x+1) - y(x)$ and

$$\tilde{F}(y, z, w) = F(y, y+z, y+2z+w).$$

Polynomial solutions

For $\mathbf{i} = (i_1, i_2, i_3) \in \mathbb{N}^3$, we define $||\mathbf{i}|| = i_1 + i_2 + i_3$. Write

$$\tilde{F} = \sum_{||\mathbf{i}||=D} c_{\mathbf{i}} y^{i_1} z^{i_2} w^{i_3},$$

where $c_{\mathbf{i}} \in \mathbb{K}$. Set

$$\begin{aligned}
\mathcal{E}(\tilde{F}) &= \{\mathbf{i} \in \mathbb{N}^3 \,|\, c_{\mathbf{i}} \neq 0\}, \\
m(\tilde{F}) &= \min\{i_2 + 2i_3 \,|\, \mathbf{i} \in \mathcal{E}(\tilde{F})\}, \\
\mathcal{M}(\tilde{F}) &= \{\mathbf{i} \in \mathcal{E}(\tilde{P}) \,|\, i_2 + 2i_3 = m(\tilde{F})\}, \\
\mathcal{P}_{\tilde{F}}(t) &= \sum_{\mathbf{i} \in \mathcal{M}(\tilde{F})} c_{\mathbf{i}} t^{i_2} [t(t-1)]^{i_3}.
\end{aligned}$$

We call $\mathcal{P}_{\tilde{F}}(t)$ the indicial polynomial of $\tilde{F}$ (at infinity).

Polynomial solutions

Proposition 4: Let $\mathcal{P}_{\tilde{F}}(t)$ be the indicial polynomial of $\tilde{F}$ at infinity. Then $\mathcal{P}_{\tilde{F}}(t) \neq 0$.

Theorem 4 (Vo-Z. 2020): Let $p(x)$ be a nonzero polynomial solution of $\tilde{F}(y(x), \Delta y(x), \Delta^2 y(x)) = 0$ with degree d. Then $\mathcal{P}_{\tilde{F}}(d) = 0$.

Algorithms

Algorithm 2: Given a separable AOΔE $\frac{P_1(y(x+1))}{Q_1(y(x+1))} = \frac{P_2(y(x))}{Q_2(y(x))}$ with $\gcd(P_i, Q_i) = 1$ and $\deg \frac{P_1}{Q_1} = \deg \frac{P_2}{Q_2} \geq 1$, $i = 1, 2$. Compute a degree bound for its rational solutions.

(1) Let $\tilde{P}_j(z, w) = w^n P_j\left(\frac{z}{w}\right)$ and $\tilde{Q}_j(z, w) = w^n Q_j\left(\frac{z}{w}\right)$, $j = 1, 2$. Consider

$$\begin{cases} \tilde{P}_1(A(x+1), B(x+1)) = \tilde{P}_2(A(x), B(x)), \\ \tilde{Q}_1(A(x+1), B(x+1)) = \tilde{Q}_2(A(x), B(x)), \end{cases} \tag{18}$$

where A, B are unknown. Derive the following nonzero AOΔEs for $A(x)$ and B(x) from (18) by using Algorithm 1:

$$F_A(A(x), A(x+1), A(x+2)) = 0, F_B(B(x), B(x+1), B(x+2)) = 0.$$

Algorithms

(2) Determine the indicial polynomials $\mathcal{P}_{F_A}$ and $\mathcal{P}_{F_B}$ of F_A and F_B, respectively. Let

$$D_A = \{\text{non-negative integer solutions of } \mathcal{P}_{F_A}(t)\},$$
$$D_B = \{\text{non-negative integer solutions of } \mathcal{P}_{F_B}(t)\}.$$

Return $\max(D_A \cup D_B)$.

Algorithms

Algorithm 3: Given an irreducible autonomous first-order AOΔE $F(y(x), y(x+1)) = 0$. Compute a non-constant rational solution or return "NULL".

(1) If $\deg_y(F) \neq \deg_z(F)$, then output "NULL". Otherwise, go to step 2.

(2) Compute the genus g of the corresponding curve $\mathcal{C}_F$ defined by $F(y, z) = 0$. If $g \neq 0$, then output "NULL". Otherwise, go to step 3.

(3) Using Vo-Grasegger-Winkler's algorithm, determine an optimal parametrization for $\mathcal{C}_F$, say $\mathcal{P}(t) = (p_1(t), p_2(t))$.

Algorithms

(4) Apply Algorithm 2 to compute a degree bound N for rational solutions of the separable AOΔE $p_1(y(x+1)) = p_2(y(x))$.

(5) Set $M = N \cdot \deg p_1$. Use Feng-Gao-Huang's algorithm to determine a non-constant rational solution of $F(y) = 0$ whose degree is at most M. Return the rational solution if there is any. Otherwise, return "NULL".

Example

Consider the first-order autonomous AOΔE:

$$F = (12y(x)+49)y(x+1)^2 - (12y^2+62y+56)y(x+1) + y(x)^2 + 8y(x) + 16 = 0. \quad (19)$$

It is clear that $\deg_y(F) = \deg_z(F) = 2$. The corresponding algebraic curve is of genus zero and it has an optimal parametrization

$$\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{9t^2 - 12t + 4}{12t}, \frac{9t^2 + 36t + 4}{12(t+4)} \right).$$

Using the above parametrization, we can derive the following associated separable AOΔE of (19):

$$\frac{9y(x+1)^2 - 12y(x+1) + 4}{y(x+1)} = \frac{9y(x)^2 + 36y(x) + 4}{y(x)+4}. \quad (20)$$

Example

Using Algorithm 2, we see that the degree bound for rational solutions of (20) is 2. Thus, the degrees of rational solutions of $F(y) = 0$ are bounded by 4. Applying Feng-Gao-Huang's algorithm, we determine a rational solution, say

$$y(x) = \frac{(1 - 4x + 2x^2)^2}{2x(1 - 3x + 2x^2)}.$$

Conclusion

- An algebraic geometric approach for studying rational solutions of first-order AOΔEs.
- A degree bound for rational solutions of autonomous first-order AOΔEs, and thus derive a complete algorithm for computing corresponding rational solutions.

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Thanks!