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Rational Solutions of High-Order Algebraic Ordinary Differential Equations*

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Abstract This paper considers algebraic ordinary differential equations (AODEs) and study their polynomial and rational solutions. The authors first prove a sufficient condition for the existence of a bound on the degree of the possible polynomial solutions to an AODE. An AODE satisfying this condition is called noncritical. Then the authors prove that some common classes of low-order AODEs are noncritical. For rational solutions, the authors determine a class of AODEs, which are called maximally comparable, such that the possible poles of any rational solutions are recognizable from their coefficients. This generalizes the well-known fact that any pole of rational solutions to a linear ODE is contained in the set of zeros of its leading coefficient. Finally, the authors develop an algorithm to compute all rational solutions of certain maximally comparable AODEs, which is applicable to 78.54% of the AODEs in Kamke's collection of standard differential equations.

Keywords Algebraic ordinary differential equations, algorithms, polynomial solutions, rational solutions.

1 Introduction

An algebraic ordinary differential equation (AODE) is of the form

$$F(x, y, y', \cdots, y^{(n)}) = 0,$$

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where F is a polynomial in $y, y', \cdots, y^{(n)}$ with coefficients in $\mathbb{K}(x)$, the field of rational functions over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $n \in \mathbb{N}$. For instance, $\mathbb{K}$ can be the field of complex numbers, or the field of algebraic numbers. AODEs arise in many areas of applications, such as physics, combinatorics and statistics. Therefore, being able to effectively determine (closed form) solutions of a given AODE is one of the central problems in mathematics, computer science, and their applications.

Although linear ODEs[1] have been intensively studied, there are still many challenging problems in solving (nonlinear) AODEs. General approaches for solving AODEs are only available for very specific subclasses. For example, Riccati equations, which have the form $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ for some $f_0, f_1, f_2 \in \mathbb{K}(x)$, can be considered as the simplest form of nonlinear AODEs. In [2], Kovacic gave a complete algorithm for determining algebraic solutions of a Riccati equation with rational function coefficients. The study of general solutions without movable singularities can be found in [3–5] for first-order, and in [1, 6] for higher-order AODEs.

The problem of finding all solutions of an arbitrary AODE is very difficulty in general, but it is natural to ask whether a given AODE admits some special kinds of solutions. In [7], Eremenko gave a theoretical bound for the degree of rational solutions of a first-order AODE. This reduces the problem of finding all rational solutions of a first-order AODE to finding all solutions of a system of algebraic equations in the coefficients of a possible rational solution.

During the last two decades, an algebraic geometric method for AODEs has been developed (see [8–12]). The main idea of this approach is based on considering of the AODE as an algebraic equation in the dependent variable and its derivatives. This algebraic equation defines an algebraic hypersurface in a suitable affine space. Therefore, tools from algebraic geometry are applicable. Using this idea, Feng and Gao proposed an algorithm for computing a rational general solution of an autonomous first-order AODE[8, 9]. The non-autonomous cases are studied in [10] and are completed in [12]. In contrast, general approaches to treat higher order AODEs are scant in the literature. In [13], the authors generalized the algebraic geometric method mentioned above to arbitrary order AODEs whose associated hypersurfaces are given together with proper parametrizations and studied their rational general solutions.

AODEs having no rational general solution can still have several particular rational solutions. The algorithms developed in [13] are not able to detect these solutions. Therefore, it is necessary to develop new algorithms for determining all rational solutions, including both general and particular solutions.

In this paper, we study the properties of the possible polynomial and rational solutions of an AODE of arbitrary order. We prove a sufficient condition for the existence of a bound on the degree of the possible polynomial solutions to a given AODE. An AODE satisfying this condition is called noncritical. Besides, we develop algorithms to test the noncritical property of an AODE, and if so, find such a degree bound. The easy determination of the condition allows us to confirm that some common classes of low-order AODEs are noncritical (see Theorems 3.6 and 3.7). This result can be considered as a refinement of the works of Krushel'nitskij in [14], and Cano in [15] concerning polynomial solutions.

It is well-known that the set of poles of rational solutions to a linear ODE with polynomial coefficients is contained in the set of zeros of its highest coefficient. This fact allows us to easily recognize possible poles of a rational solutions from the coefficients of a given linear AODE. Unfortunately, this fact does not hold for nonlinear AODEs. However, we show that there is a large subclass of AODEs for which this fact is still valid. In order to do that, we equip the set of monomials in the unknown y and its derivatives with a suitable partial order (see Definition 4.1). If an AODE admits a highest monomial with respect to this ordering, then the poles of its possible rational solutions can only occur at the zeros of the corresponding highest coefficient (Theorem 4.3). This generalizes the same fact of linear AODEs to the nonlinear ones. An AODE satisfying the existence of the highest monomial is called maximally comparable.

The notion of maximally comparable AODEs already appears in [16], where the authors considered first-order AODEs only. The authors proved that for every maximally comparable first-order AODE, there is a finite upper bound on the degrees of its rational solutions, and developed an algorithm to determine such a bound. Here, we extend this notion to higher order AODEs. Unlike in the first-order cases, there might not exist a priori bound on the degree of a possible rational solution for the higher order ones. We define a class of AODEs, called completely maximally comparable, for which the existence of an upper bound for its rational solutions is guaranteed. The class of maximally comparable AODEs covers 78.54% AODEs from a standard collection by Kamke[17], and all of these are in fact also completely maximally comparable. This suggests that completely maximally comparable AODEs, which are in the scope of our algorithm for determining all rational solutions (see Algorithm 4.7), form a large subclass of those AODEs that actually arise in practice.

The rest of the paper is organized as follows. Section 2 is devoted to a study of order bounds for the poles of a Laurent series solution to an AODE. In Section 3 we give a sufficient condition for an AODE to have a degree bound for its polynomial solutions. We also prove that some common classes of low-order AODEs satisfy this condition. Rational solutions of maximally comparable AODEs are considered in Section 4. Finally, we perform a statistical investigation with a collection of AODEs from a standard textbook by Kamke[17].

2 An Order Bound for Laurent Series Solutions

This section can be considered as an alternative interpretation of the Newton polygon method for AODEs, specifically tailored to Laurent series solutions. In particular, given an AODE, we show in Proposition 2.3 that the orders of its Laurent series solutions at any given point can be bounded in an algorithmic way. The proposition yields an easy determination of the bound. More general constructions that apply to wider classes of series solutions can be found in [15, 18, 19].

Given $x_0 \in \mathbb{K} \cup \{\infty\}$, a Laurent series f at $x = x_0$ has the form

$$\begin{aligned} &\sum_{k=m}^{\infty} c_k (x - x_0)^k, \qquad && \text{if } \ x_0 \in \mathbb{K}, \\ &\sum_{k=m}^{\infty} c_k x^{-k}, && \text{if } \ x_0 = \infty, \end{aligned}$$

where $c_k \in \mathbb{K}, c_m \neq 0$ and $m \in \mathbb{Z}$. We call $-m$ the order of f (at $x = x_0$), and denote it by $\mathrm{ord}_{x_0}(f)$. The coefficient c_m is called the lowest coefficient of f (at $x = x_0$), and denoted by $c_{x_0}(f)$. Then we can rewrite f as follows:

$$\begin{aligned} &c_{x_0}(f)(x - x_0)^{-\mathrm{ord}_{x_0}(f)} && + \quad \text{higher order terms in } (x - x_0), && \text{if } \ x_0 \in \mathbb{K}, \\ &c_\infty(f) x^{\mathrm{ord}_\infty(f)} && + \quad \text{lower order terms in } x, && \text{if } \ x_0 = \infty. \end{aligned}$$

For each $I = (i_0, i_1, \cdots, i_n) \in \mathbb{N}^{n+1}$ and $r \in \{0, 1, \cdots, n\}$, we set $||I||_r = i_r + \cdots + i_n$. For $r = 0$ we write $||I||$ instead of $||I||_0$. We also define $||I||_\infty = i_1 + 2i_2 + \cdots + n i_n$.

Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I(x) y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n} \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. We will use the following notations:

$$\begin{aligned} \mathcal{E}(F) &= \{I \in \mathbb{N}^{n+1} \mid f_I \neq 0\}, \\ d(F) &= \max\{||I|| \mid I \in \mathcal{E}(F)\}, \\ \mathcal{D}(F) &= \{I \in \mathcal{E}(F) \mid ||I|| = d(F)\}. \end{aligned}$$

Moreover, for each $x_0 \in \mathbb{K}$, we denote

$$\begin{aligned} m_{x_0}(F) &= \max\{\mathrm{ord}_{x_0} f_I + ||I||_\infty \mid I \in \mathcal{D}(F)\}, \\ \mathcal{M}_{x_0}(F) &= \{I \in \mathcal{D}(F) \mid \mathrm{ord}_{x_0} f_I + ||I||_\infty = m_{x_0}(F)\}, \\ \mathcal{P}_{x_0,F}(t) &= \sum_{I \in \mathcal{M}_{x_0}(F)} c_{x_0}(f_I) \cdot \prod_{r=0}^{n-1} (-t - r)^{||I||_{r+1}}, \end{aligned}$$

and if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, we set

$$b_{x_0}(F) = \max\left\{ \frac{\mathrm{ord}_{x_0} f_I + ||I||_\infty - m_{x_0}(F)}{d(F) - ||I||} \;\Big|\; I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\}.$$

In case that $x_0 = \infty$, we also denote

$$\begin{aligned} m_\infty(F) &= \max\{\mathrm{ord}_\infty f_I - ||I||_\infty \mid I \in \mathcal{D}(F)\}, \\ \mathcal{M}_\infty(F) &= \{I \in \mathcal{D}(F) \mid \mathrm{ord}_\infty f_I - ||I||_\infty = m_\infty(F)\}, \\ \mathcal{P}_{\infty,F}(t) &= \sum_{I \in \mathcal{M}_\infty(F)} c_\infty(f_I) \cdot \prod_{r=0}^{n-1} (t - r)^{||I||_{r+1}}, \end{aligned}$$

and

$$b_\infty(F) = \max\left\{ \frac{\mathrm{ord}_\infty f_I - ||I||_\infty - m_\infty(F)}{d(F) - ||I||} \;\Big|\; I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\},$$

if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$.

Definition 2.1 Let $F(y) \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. For each $x_0 \in \mathbb{K} \cup \{\infty\}$, we call $\mathcal{P}_{x_0,F}$ the indicial polynomial of F at $x = x_0$.

Remark 2.2 The term "indicial polynomial" already appears in the literature, for instance in [1, 20], when linear ODEs are considered. Definition 2.1 can be seen as a generalization of this term to non-linear AODEs.

The following proposition shows the relation between the order of a Laurent series solution and the indicial polynomial.

Proposition 2.3 *Given an AODE $F(y) = 0$, and $x_0 \in \mathbb{K} \cup \{\infty\}$. If $r \geq 1$ is the order of a Laurent series solution of $F(y) = 0$ at $x = x_0$, then exactly one of the following properties holds:*

(i) $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, *and* $r \leq b_{x_0}(F)$;

(ii) r *is a positive integer root of* $\mathcal{P}_{x_0,F}(t)$.

Proof Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I(x) y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n} \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. Let $x_0 \in \mathbb{K}$ and $z \in \mathbb{K}((x - x_0)) \setminus \mathbb{K}$ be a Laurent series solution of $F(y) = 0$ of order $r \geq 1$. Then $z^{(k)}$ is of order $k + r$ for each $k \in \mathbb{N}$. For each $I \in \mathcal{E}(F)$, we may write the coefficient f_I in the following form:

$$f_I = \frac{c_{x_0}(f_I)}{(x - x_0)^{\operatorname{ord}_{x_0} f_I}} + h_I,$$

where $h_I \in \mathbb{K}((x))$ and $\operatorname{ord}_{x_0} h_I < \operatorname{ord}_{x_0} f_I$. Since z is a solution of $F(y) = 0$, we have

$$\begin{aligned} 0 &= F(z) \\ &= S_1 + S_2 + S_3 + S_4, \end{aligned}$$

where

$$S_1 = \sum_{I \in \mathcal{M}_{x_0}(F)} \frac{c_{x_0}(f_I)}{(x - x_0)^{\operatorname{ord}_{x_0} f_I}} \cdot z^{i_0} (z')^{i_1} \cdots (z^{(n)})^{i_n}, \qquad S_2 = \sum_{I \in \mathcal{M}_{x_0}(F)} h_I \cdot z^{i_0} (z')^{i_1} \cdots (z^{(n)})^{i_n},$$
$$S_3 = \sum_{I \in \mathcal{D}(F) \setminus \mathcal{M}_{x_0}(F)} f_I z^{i_0} (z')^{i_1} \cdots (z^{(n)})^{i_n}, \qquad S_4 = \sum_{I \in \mathcal{E}(F) \setminus \mathcal{D}(F)} f_I z^{i_0} (z')^{i_1} \cdots (z^{(n)})^{i_n}.$$

The order of each term in S_1 is equal to $D = d(F)r + m_{x_0}(F)$, which is strictly larger than that of each term in S_2 and S_3. One of the following two cases will occur:

Case 1 The order of S_1 is equal to D. Then the terms of order D in S_1 must be killed by the terms of order D in S_4. In this case, we have $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$. By comparing the orders of terms in S_4, we obtain

$$D \leq \max \{ ||I|| \cdot r + ||I||_\infty + \operatorname{ord}_{x_0} f_I \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \} .$$

On the other hand, since $D = d(F)r + m_{x_0}(F)$, we conclude that

$$r \leq \max \left\{ \frac{||I||_\infty + \operatorname{ord}_{x_0} f_I - m_{x_0}(F)}{d(F) - ||I||} \;\Big|\; I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\} .$$

In other words, $r \le b_{x_0}(F)$.

Case 2 The order of S_1 is strictly smaller than D. For each $k \in \mathbb{N}$, a direct computation shows that the lowest coefficient $z^{(k)}$ at $x = x_0$ is

$$c_{x_0}(z^{(k)}) = c_{x_0}(z) \prod_{s=1}^{k} (-r - s + 1).$$

Therefore, the lowest coefficient of the term indexed by $I \in \mathcal{M}_{x_0}(F)$ in S_1 is

$$c_{x_0}(f_I) \cdot \prod_{k=0}^{n} \left(c_{x_0}(z) \prod_{s=1}^{k} (-r - s + 1) \right)^{i_k} = c_{x_0}(f_I) c_{x_0}(y)^{||I||} \prod_{s=1}^{n} (-r - s + 1)^{||I||_s}.$$

Since the orders of terms in S_1 are the same and they are strictly larger than that of S_1, the sum of those lowest coefficients must be zero. In other words, we have

$$\sum_{I \in \mathcal{M}_{x_0}(F)} c_{x_0}(f_I) c_{x_0}(y)^{||I||} \prod_{s=1}^{n} (-r - s + 1)^{||I||_s} = 0.$$

The left side of the above equality is exactly $c_{x_0}(y)^{d(F)} \cdot \mathcal{P}_{x_0,F}(r)$. Hence, r is a positive integer root of $\mathcal{P}_{x_0,F}(r)$.

The proof in the case $x_0 = \infty$ is analogous. ∎

Remark 2.4 Proposition 2.3 shows that the indicial polynomial might provide information about the order of a Laurent series solution.

1) For a linear homogeneous ordinary differential equation $F(y) = 0$, since $\mathcal{E}(F) = \mathcal{D}(F)$, Case 1 in the proof of Proposition 2.3 never happens. In this case, the indicial polynomial $\mathcal{P}_{x_0,F}$ is nonzero (see Theorem 3.6), and the degree of a Laurent series solution is always a zero of the indicial polynomial. This is a well-know fact in linear ODEs.

2) If $\mathcal{E}(F) = \mathcal{D}(F)$ and the indicial polynomial $\mathcal{P}_{x_0,F}(t)$ is identically zero, then Proposition 2.3 does not give any information about the order bound of Laurent series solution of $F(y) = 0$ at $x = x_0$. For instance, for the differential equation in Example 3.5 below, the order of a Laurent series solution at infinity can be arbitrarily large.

3 Polynomial Solutions of Noncritical AODEs

In [14], Krushel'nitskij discussed the properties of the degree of a polynomial solution for a given AODE. By using the Newton polygon at infinity, Cano proposed an algorithm for determining a bound for the degrees of polynomial solutions of an AODE provided that the Newton polygon of the given AODE satisfies certain additional assumptions (see [15, Section 2.2]). Whenever a degree bound is found, one can determine all polynomial solutions by the undeterminate coefficient method. However, to the best of our knowledge, no full algorithm for computing all polynomial solutions of AODEs exists so far.

In this section, we use Proposition 2.3 to give a sufficient condition (see Definition 3.1) for the existence of a bound for the degrees of polynomial solutions. We prove that several common classes of AODEs satisfy this sufficient condition (see Theorem 3.6 and Theorem 3.7). Furthermore, we will show in Section 5 that all AODEs in Kamke's collection[17] satisfy this condition.

Definition 3.1 An AODE $F(y)=0$ is called noncritical if $\mathcal{P}_{\infty,F}(t)\neq 0$.

Corollary 3.2 *If an AODE $F(y)=0$ is noncritical, then there exists a bound on the degree of any polynomial solution.*

Proof Straightforward from Proposition 2.3. ∎

Algorithm 3.3 Given a noncritical AODE $F(y)=0$, compute all its polynomial solutions.

1) Compute $\mathcal{P}_{\infty,F}(t)$. If $\mathcal{P}_{\infty,F}(t)$ has integer roots, then set r_1 to be the largest integer root. Otherwise, set $r_1=0$.

2) Compute $r_2=\lfloor b_\infty(F)\rfloor$ if $\mathcal{E}(F)\setminus\mathcal{D}(F)\neq\emptyset$. Otherwise set $r_2=0$.

3) Set $r=\max\{r_1,r_2,0\}$. Make an ansatz $z=\sum_{i=0}^{r}c_ix^i$, where the c_i's are unknown. Substitute z into $F(y)=0$ and solve the corresponding algebraic equations by using Gröbner bases.

4) Return the solutions from the above step.

The termination of Algorithm 3.3 is clear. The correctness follows from Proposition 2.3.

Example 3.4 (see [17]) Consider the differential equation:

$$F(y)=a^2y^2y''^2-2a^2yy'^2y''+a^2y'^4-b^2y''^2-y'^2=0, \tag{1}$$

where $a,b\in\mathbb{K}$ and $a\neq 0$. The following table is a list of the exponents of terms of F and related information.

From Table 1 we see that $\mathcal{D}(F)$ is the set of exponents in the first three lines, and $\mathcal{E}(F)\backslash\mathcal{D}(F)$ is the set of exponents in the last two lines. A direct computation shows that $m_\infty(F)=-4$, $\mathcal{M}_\infty(F)=\mathcal{D}(F)$, and $\mathcal{P}_{\infty,F}(t)=a^2t^2\neq 0$. Therefore, the differential equation (1) is noncritical. Furthermore, we find that $b_\infty(F)=1$.

Table 1

$I\in\mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	f_I
$(2,0,2)$	4	4	a^2
$(1,2,1)$	4	4	$-2a^2$
$(0,4,0)$	4	4	a^2
$(0,0,2)$	2	4	$-b^2$
$(0,2,0)$	2	2	-1

By Proposition 2.3, every polynomial solution of (1) has degree at most 1. By making an ansatz and solving the corresponding algebraic equations, we obtain all polynomial solutions, which are c, $c+\frac{x}{a}$, and $c-\frac{x}{a}$, where c is an arbitrary constant in $\mathbb{K}$.

To the best of our knowledge, almost every AODE in the literature is noncritical. But a few of these fail to be noncritical; below is an example.

Example 3.5 Let us consider the following differential equation:

$$F(y) = xyy'' - xy'^2 + yy' = 0.$$

This differential equation appeared in [7], and has been studied by means of the Newton polygon method in [19]. It admits the polynomial solution $y(x) = cx^n$ for arbitrary $c \in \mathbb{K}$ and $n \in \mathbb{N}$. A direct computation shows that its indicial polynomial at infinity is zero. So $F(y) = 0$ is a critical AODE.

We show in the next two theorems that most common classes of low order AODEs are noncritical.

Theorem 3.6 *Let $\mathcal{L} \in \mathbb{K}(x)\left[\frac{\partial}{\partial x}\right]$ be a differential operator, and $P(x, y, z) \in \mathbb{K}(x)[y, z]$ a polynomial in two variables with coefficients in $\mathbb{K}(x)$. Then for each $n > 0$, the differential equation $\mathcal{L}(y) + P(x, y, y^{(n)}) = 0$ is noncritical.*

In particular, linear AODEs, first-order AODEs (which have the form $F(x, y, y') = 0$ for some $F \in \mathbb{K}(x)[y, y']$), and quasi-linear second-order AODEs (which have the form $y'' + G(x, y, y') = 0$ for some $G \in \mathbb{K}(x)[y, y']$), are noncritical.

Proof Let $F(y) = \mathcal{L}(y) + P(x, y, y^{(n)})$. We prove that $\mathcal{P}_{\infty,F}$ is nonzero.

First, we consider the case that P is a linear polynomial in y and z. Then F is a linear differential polynomial, say

$$F(y) = f_{I_{-1}} + f_{I_0} y + \cdots + f_{I_m} y^{(m)},$$

where $f_{I_i} \in \mathbb{K}(x)$ and $f_{I_m} \neq 0$ and $m \in \mathbb{N}$. A direct computation shows that the indicial polynomial of F at infinity is of the form

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{i=0,1,\cdots,m \\ I_i \in \mathcal{M}_\infty(F)}} c_\infty(f_{I_i}) \cdot \prod_{s=1}^{i} (t - s + 1),$$

which is a nonzero polynomial. Therefore, linear AODEs are noncritical.

Next, assume that P is of total degree at least 2. Then we have $\mathcal{D}(F) = \mathcal{D}(P(x, y, y^{(n)}))$ and $\mathcal{M}_\infty(F) = \mathcal{M}_\infty(P(x, y, y^{(n)}))$. We write $P(x, y, y^{(n)})$ in the form

$$P(x, y, y^{(n)}) = \sum_{(i,j) \in \mathbb{N}^2} f_{i,j}(x) y^i (y^{(n)})^j.$$

Then $\mathcal{M}_\infty(F)$ consists of elements of the form $e_{i,j} = (i, 0, \cdots, 0, j) \in \mathbb{N}^{n+1}$. A direct calculation reveals that

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{j=1,2,\cdots,n \\ e_{i,j} \in \mathcal{M}_\infty(F)}} c_\infty(f_{i,j}) \cdot [t(t-1)\cdots(t-n+1)]^j.$$

The indicial polynomial $\mathcal{P}_{\infty,F}(t)$ can be viewed as the evaluation of the nonzero univariate polynomial

$$g(T) = \sum_{\substack{j=1,2,\cdots,n\\ e_{i,j}\in\mathcal{M}_\infty(F)}} c_\infty(f_{i,j})\cdot T^j \quad \text{at} \quad T = t(t-1)\cdots(t-n+1).$$

On the other hand, since $t(t-1)\cdots(t-n+1)$ is transcendental over $\mathbb{K}$, we conclude that $\mathcal{P}_{\infty,F} \neq 0$. ∎

Theorem 3.7 *Let $\mathcal{L} \in \mathbb{K}(x)\left[\frac{\partial}{\partial x}\right]$ be a differential operator with coefficients in $\mathbb{K}(x)$, and $Q(y,z,w) \in \mathbb{K}[y,z,w]$ a polynomial in three variables with coefficients in $\mathbb{K}$. Then for each $m,n>0$, the differential equation $\mathcal{L}(y) + Q(y, y^{(n)}, y^{(m)}) = 0$ is noncritical.*

In particular, autonomous second-order AODEs (which have the form $F(y,y',y'') = 0$ for some $F \in \mathbb{K}[y,y',y'']$), and quasi-linear autonomous third-order AODEs (which have the form $y''' + G(y,y',y'') = 0$ for some $G \in \mathbb{K}[y,y',y'']$), are noncritical.

Proof Let $F(y) = \mathcal{L}(y) + Q(y, y^{(m)}, y^{(n)})$. Without loss of generality, we can assume that $0 < m < n$. As we have seen from the previous proposition, a linear AODE is noncritical. Therefore we can assume further that Q is of total degree at least 2. Then we have $\mathcal{D}(F) = \mathcal{D}(Q(y, y^{(m)}, y^{(n)}))$ and $\mathcal{M}_\infty(F) = \mathcal{M}_\infty(Q(y, y^{(m)}, y^{(n)}))$. Let us write $Q(y, y^{(m)}, y^{(n)})$ in the form

$$Q(y, y^{(m)}, y^{(n)}) = \sum_{(ijk)\in\mathbb{N}^3} f_{ijk} y^i (y^{(m)})^j (y^{(n)})^k.$$

For simplicity, we denote $e_{ijk} = (i,0,\cdots,0,j,0,\cdots,0,k) \in \mathbb{N}^{n+1}$, where j is the $(m+1)$-th coordinate. Then $\mathcal{M}_\infty(F)$ consists of all e_{ijk} such that $i+j+k = d(F)$ and $mj+nk = m_\infty(F)$. A direct computation implies that

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{(i,j,k)\in\mathbb{N}^3\\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk})\cdot (t(t-1)\cdots(t-m+1))^{j+k}\cdot((t-m)\cdots(t-n+1))^k.$$

This polynomial can be rewritten as:

$$\mathcal{P}_{\infty,F}(t) = A^{\frac{m_\infty(F)}{m}} \cdot \sum_{\substack{k=0,1,\cdots,n\\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk}) \left(\frac{B}{A^{\frac{(n-m)}{m}}}\right)^k, \tag{2}$$

where $A = t(t-1)\cdots(t-m+1)$ and $B = (t-m)\cdots(t-n+1)$. The sum in (2) can be viewed as the evaluation of the univariate polynomial

$$h(T) = \sum_{\substack{k=0,1,\cdots,n\\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk})T^k \quad \text{at} \quad T = \frac{B}{A^{\frac{(n-m)}{m}}}.$$

Since the projection which maps e_{ijk} to k is injective, we have that $h(T)$ is nonzero. On the other hand, since $\frac{B}{A^{\frac{(n-m)}{m}}}$ is transcendental over $\mathbb{K}$, we conclude that $\mathcal{P}_{\infty,F}$ is nonzero. ∎

4 Rational Solutions of Maximally Comparable AODEs

A rational function admits a Laurent series expansion at every point in $\mathbb{K}$ or at infinity. A point $x_0 \in \mathbb{K} \cup \{\infty\}$ is called a pole of a rational function if the order of the Laurent series expansion of the rational function is positive. In this case, the order of the Laurent series expansion is called the order of the pole. A rational function has only finitely many poles, and the order at each pole is a positive integer.

It is well-known that poles of rational solutions of a linear ODE with polynomial coefficients only occur at the zeros of the highest coefficient of the equation (see [1]). This fact does not hold for nonlinear AODEs in general. In this section, we describe a class of AODEs for which the above fact is still true. In order to do that, we first need to define what is the "highest" coefficient in the nonlinear case. To do so, we equip the set of monomials in y and its derivatives with a suitable partial order (see Definition 4.1). We show in Theorem 4.3 that if the given AODE has a greatest monomial with respect to this ordering, then the poles of its rational solutions can only occur at the zeros of the corresponding coefficient. Together with Proposition 2.3, we give a sufficient condition for such AODEs to have bounds for the orders of their poles. Therefore for AODEs satisfying this condition it is possible to find all rational solutions.

Definition 4.1 Assume that $n \in \mathbb{N}$. For each $I, J \in \mathbb{N}^{n+1}$, we say that $I \gg J$ if $||I|| \geq ||J||$ and $||I|| + ||I||_\infty > ||J|| + ||J||_\infty$.

It is straightforward to verify that the order defined as above is a strict partial ordering on $\mathbb{N}^{n+1}$, i. e., the following properties hold for all $I, J, K \in \mathbb{N}^{n+1}$:

(i) irreflexivity: $I \not\gg I$;

(ii) transitivity: if $I \gg J$ and $J \gg K$, then $I \gg K$;

(iii) asymmetry: if $I \gg J$, then $J \not\gg I$.

For $I, J \in \mathbb{N}^{n+1}$, we say that I and J are comparable if either $I \gg J$ or $J \gg I$. Otherwise, they are called incomparable. It is clear that the order $\gg$ is not a total order on $\mathbb{N}^{n+1}$. For example, $(2, 0)$ and $(0, 1)$ are incomparable. For a given point I in $\mathbb{N}^{n+1}$, it is straightforward to verify that the number of points that are incomparable to I is finite.

Let S be a subset of $\mathbb{N}^{n+1}$. An element $I \in S$ is called a greatest element of S if $I \gg J$ for every $J \in S \setminus \{I\}$. By the asymmetry property of $\gg$, the set S has at most one greatest element. This motivates the following definition.

Definition 4.2 An AODE $F(y) = 0$ is called maximally comparable if $\mathcal{E}(F)$ admits a greatest element with respect to $\gg$. In this case, the corresponding monomial is called the highest monomial, and the coefficient of the highest monomial is called the highest coefficient.

The term maximally comparable already appeared in [16]. In [16, Section 3], the authors defined maximally comparable first-order AODEs and studied their rational solutions. The authors also showed that most first-order AODEs are maximally comparable. Here, we extend the work of [16] to higher-order AODEs. We will see later that most high-order AODEs in the literature are also maximally comparable. The following theorem can be viewed as a generalization of [16, Theorem 3.4].

Theorem 4.3 *Let $F(y) = \sum_{I\in\mathbb{N}^{n+1}} f_I y^{i_0}(y')^{i_1}\cdots(y^{(n)})^{i_n} \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n > 0$. Assume that $F(y) = 0$ is maximally comparable, and I_0 is the greatest element of $\mathcal{E}(F)$ with respect to $\gg$. Then the poles of a rational solution of $F(y) = 0$ can only occur at infinity or at the zeros of $f_{I_0}(x)$.*

Proof We prove the above claim by contradiction. Suppose that there is $x_0 \in \mathbb{K}$ such that x_0 is a pole of order $r \geq 1$ of a rational solution of the AODE $F(y) = 0$, and $f_{I_0}(x_0) \neq 0$. Then $\mathrm{ord}_{x_0} f_{I_0} = 0$.

We first prove that $\mathcal{M}_{x_0}(F) = \{I_0\}$. Since I_0 is the greatest element of $\mathcal{E}(F)$ with respect to $\gg$, we see that $||I_0|| \geq ||J||$ for all $J \in \mathcal{E}(F)$. So $I_0 \in \mathcal{D}(F)$. Now let us fix any $J \in \mathcal{D}(F)\setminus\{I_0\}$. Since $||I_0|| = ||J||$ and $||I_0|| + ||I||_\infty > ||J|| + ||J||_\infty$, we have that $||I_0||_\infty > ||J||_\infty$. Therefore, we conclude that $\mathrm{ord}_{x_0}(f_{I_0}) + ||I_0||_\infty > \mathrm{ord}_{x_0}(f_J) + ||J||_\infty$ because $\mathrm{ord}_{x_0} f_{I_0} = 0 \geq \mathrm{ord}_{x_0}(f_J)$. In other words, I_0 is the only element of $\mathcal{M}_{x_0}(F)$.

Since $\mathcal{M}_{x_0}(F) = \{I_0\}$, the indicial polynomial at $x = x_0$ has the form

$$\mathcal{P}_{x_0,F}(t) = c_{x_0}(f_{I_0}) \cdot \prod_{r=0}^{n-1} (-t-r)^{||I_0||_{r+1}}.$$

It is straightforward to see that $\mathcal{P}_{x_0,F}(t)$ has no positive integer root. Due to Proposition 2.3 and $r \geq 1$, we have $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$ and

$$\begin{aligned} r \leq b_{x_0}(F) &= \max\left\{\frac{\mathrm{ord}_{x_0}(f_J) + ||J||_\infty - ||I_0||_\infty}{||I_0|| - ||J||} \;\middle|\; J \in \mathcal{E}(F)\setminus\mathcal{D}(F)\right\} \\ &= \max\left\{1 - \frac{-\mathrm{ord}_{x_0}(f_J) + (||I_0|| + ||I_0||_\infty) - (||J|| + ||J||_\infty)}{||I_0|| - ||J||} \;\middle|\; J \in \mathcal{E}(F)\setminus\mathcal{D}(F)\right\} \\ &< 1. \end{aligned}$$

This contradicts the assumption that $r \geq 1$. ∎

The above theorem implies that for maximally comparable AODEs, there are only finitely many candidates for poles of rational function solutions. Moreover, the poles of those rational solutions, if there are any, can only occur at the zeros of the highest coefficient with respect to the partial order $\gg$, or at infinity. This can be considered as a generalization to nonlinear AODEs of the same fact for linear ordinary differential equations. Once a candidate for a pole of a rational solution is found, one may use Proposition 2.3 to bound the order at this candidate. As we mentioned in Example 3.5, Proposition 2.3 may fail to give an order bound at certain points, as the following example illustrates.

Example 4.4 Consider the following AODE:

$$F(y) = x^3yy''' + xyy' - x(y')^2 + yy' = 0.$$

It is straightforward to verify that $F(y) = 0$ is maximally comparable. By Theorem 4.3, we know that the only possible pole of a rational solution of $F(y) = 0$ is at 0. However, a direct calculation implies that $\mathcal{P}_{0,F}(t) = 0$. Therefore, we cannot bound the order of a pole at zero by using Proposition 2.3.

In order to compute rational solutions of a given maximally comparable AODEs, we impose the following property to ensure that we can bound the order of candidates for poles of its rational solutions.

Definition 4.5 Let $F(y)=0$ be a maximally comparable AODE with highest coefficient $f(x)$ with respect to $\gg$. We say that $F(y)=0$ is completely maximally comparable if $\mathcal{P}_{x_0,F}(t)$ is a non-zero polynomial for every root x_0 of $f(x)$.

The following is a sufficient condition for a maximally comparable AODE to be complete.

Proposition 4.6 *Let $F(y)=0$ be a maximally comparable AODE, and assume that $\mathcal{D}(F)$ is totally ordered with respect to the ordering $\gg$. Then $\mathcal{P}_{x_0,F}(t)\neq 0$ for every $x_0\in\mathbb{K}\cup\{\infty\}$. In particular, F is completely maximally comparable.*

Proof Assume that $x_0\in\mathbb{K}\cup\{\infty\}$. Since $F(y)=0$ is a maximally comparable AODE, for each $I,J\in\mathcal{M}_{x_0}(F)$ with $I\neq J$ we have that $||I||_\infty\neq||J||_\infty$. On the other hand, for each $I\in\mathcal{M}_{x_0}(F)$ the degree of the polynomial $\prod_{r=0}^{n-1}(-t-r)^{||I||_{r+1}}$ is exactly $||I||_\infty$. Hence, we conclude that $\mathcal{P}_{x_0,F}(t)\neq 0$. ∎

We can always give an order bound for candidates of rational solutions of completely maximally comparable AODEs by using Proposition 2.3. Combined with the partial fraction decomposition of a rational function, we present the following algorithm for determining all rational solutions of a completely maximally comparable AODE.

Algorithm 4.7 Given a completely maximally comparable AODE $F(y)=0$, compute all its rational function solutions.

1) Compute the greatest element I_0 of $\mathcal{E}(F)$ with respect to $\gg$. Compute the distinct roots $x_1,x_2,\cdots,x_m$ of $f_{I_0}(x)$ in $\mathbb{K}$.

2) For $i\in\{1,2,\cdots,m\}$, compute an order bound r_i for rational solutions of $F(y)=0$ at $x=x_i$ using Proposition 2.3. Similarly, compute the order bound N of a possible pole at infinity.

3) Make an ansatz with the partial fraction decomposition

$$z=\sum_{i=1}^{m}\sum_{j=1}^{r_i}\frac{c_{ij}}{(x-x_i)^j}+\sum_{k=0}^{N}c_i x^i\,, \tag{3}$$

   where the c_{ij} and c_i are unknown. Substitute (3) into $F(y)=0$ and solve the corresponding algebraic equations by using Gröbner bases.

4) Return the solutions from the above step.

The termination of the above algorithm follows from Proposition 2.3. The correctness follows from Theorem 4.3.

Example 4.8 Consider the differential equation

$$\begin{aligned}F(y) &= x^2(x-1)^2y''^2 + 4x^2(x-1)y'y'' - 4x(x-1)yy'' \\ &\quad +4x^2y'^2 - 8xyy' + 4y^2 - 2(x-1)y'' \\ &= 0.\end{aligned}$$

We first collect some information about the exponents of terms of $F(y)$.

In Table 2, $\mathcal{D}(F)$ consists of the first 6 elements of $\mathcal{E}(F)$, and $d(F) = 2$. Furthermore, $\mathcal{D}(F)$ is totally ordered with respect to $\gg$. Therefore F is completely maximally comparable, and the exponent $(0, 0, 2)$ is the greatest element of $\mathcal{E}(F)$.

Table 2

$I \in \mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	$\|\|I\|\| + \|\|I\|\|_\infty$	f_I
$(0,0,2)$	2	4	6	$x^2(x-1)^2$
$(0,1,1)$	2	3	5	$4x^2(x-1)$
$(1,0,1)$	2	2	4	$-4x(x-1)$
$(0,2,0)$	2	2	4	$4x^2$
$(1,1,0)$	2	1	3	$-8x$
$(2,0,0)$	2	0	2	4
$(0,1,0)$	1	1	2	$-2(x-1)$

By Theorem 4.3, the poles of a rational solution of $F(y) = 0$ can only occur at the zeros of the polynomial $x^2(x-1)^2$, which are 0 and 1, and possibly at infinity.

A simple computation based on Proposition 2.3 shows that the orders of poles of a rational solution of $F(y) = 0$ at $0, 1$, and infinity are at most $0, 1$, and 1, respectively.

Hence, we make an ansatz of the form:

$$z = \frac{c_1}{x-1} + c_2 + c_3x \quad \text{for some} \quad c_1, c_2, c_3 \in \mathbb{K}.$$

Substituting z into $F(y) = 0$ and solving the corresponding algebraic equations, we find that the rational solutions of $F(y) = 0$ are c_3x and $\frac{1}{x-1} + c_3x$, where c_3 is an arbitrary constant in $\mathbb{K}$. Note that this equation has no rational general solution. Thus the algorithm in [13] is not applicable.

5 Experimental Results

In the previous sections we introduced the classes of noncritical, maximally comparable and completely maximally comparable AODEs, and deduced properties of the possible rational solutions of such equations. In this section we carry out a statistical investigation to find out how many AODEs in Kamke's well-known collection[17] are noncritical, maximally comparable, or completely maximally comparable. The corresponding Maple worksheet is available in:

https://yzhang1616.github.io/KamkeODEs.mw.

The worksheet requires the availability of the following Maple package:

https://yzhang1616.github.io/KamkeODEs.mpl.

There are 834 AODEs in Kamke's collection. All of them are noncritical. This means that our method can be used to determine all polynomial solutions, if there are any, of each AODE from Kamke's collection. Among them, there are 655 maximally comparable AODEs ($\approx$ 78.54 %). All of the maximally comparable AODEs are also completely maximally comparable.

The class of AODEs covers around 79.66 % of the entire collection of ODEs in [17]. The remaining ODEs have coefficients involving trigonometric functions ($\sin x, \cos x, \cdots$), hyperbolic functions ($\sinh x, \cosh x, \cdots$), exponential functions e^x, logarithmic functions $\log x$, or power functions with parameters in the exponents ($x^\alpha, y^\beta, \cdots$). For integer choices of the parameters, the latter ODEs will become algebraic. More precisely, there are 35 ODEs containing parameters in the power functions. For any choice of parameters such that the corresponding ODEs are algebraic, then all of them are noncritical and 21 of them (60%) are completely maximally comparable.

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