Laurent Series Solutions of Algebraic Ordinary Differential Equations

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Algebraic ordinary differential equations (AODEs)

Let $\mathbb{K}$ be an algebraic closed field of char 0, and x be an indeterminate.

Consider the AODE:

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where F is a polynomial in $y, y', \ldots, y^{(n)}$ with coeffs in $\mathbb{K}(x)$ and $n \in \mathbb{N}$ is called the order of F. We also simply write (1) as $F(y) = 0$.

Example 1. Consider the Riccati equation:

$$y' = 1 + y^2.$$

Background and motivation

Goal: Given an AODE $F(y) = 0$, find $z = \sum_{i=-r}^{\infty} c_i x^i \in \mathbb{K}((x))$ s.t.

$$F(z) = 0,$$

where r is called the order of z, and denoted as $\operatorname{ord}(z)$.

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Feng and Gao (2006): an algorithm for computing Laurent series sols at $x = \infty$ for first-order autonomous AODEs with nontrivial rational sols.

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Our contribution: Construct an order bound for Laurent series sols of arbitrary order AODEs and give a method to compute them.

General idea

Let $F(y) = 0$ be an AODE, and $m \in \mathbb{N}$.

Assume that $z \in \mathbb{K}((x))$ is a sol of $F(y) = 0$.

1. Derive an order bound B for the order of z.
2. Substitute $z = \frac{1}{x^B} w$ with $w \in \mathbb{K}[[x]]$ into $F(y) = 0$ and get a new AODE
$$G(w) = 0. \tag{2}$$
3. Compute formal power series sols of (2) with the form:
$$w = c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} + \mathcal{O}(x^m).$$
4. Return $\frac{1}{x^B} w$.

General idea

Example 2. Consider the AODE:

$$F(y) = xy' + x^2y^2 + y - 1 = 0.$$

Assume that $z \in \mathbb{K}((x))$ is a sol of $F(y) = 0$.

1. An order bound for the order of z is 2.
2. Substitute $z = \frac{1}{x^2}w$ with $w \in \mathbb{K}[[x]]$ into $F(y) = 0$ and get a new AODE

$$G(w) = xw' + w^2 - w - x^2 = 0. \tag{3}$$

3. Compute formal power series sols of (3) with the form:

$$w = 1 + 0x + \frac{1}{3}x^2 + 0x^3 - \frac{1}{45}x^4 + \mathcal{O}(x^5).$$

4. Return $\frac{1}{x^2}w$.

Outline

- Computing formal power series solutions
- Order bound for Laurent series solutions
- Applications
  - Polynomial solutions of noncritical AODEs
  - Rational solutions of maximally comparable AODEs
- Conclusion

Formal power series solutions

Let $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \ldots]$ be the ring of differential polynomials over $\mathbb{K}(x)$, where $(y^{(n)})' = y^{(n+1)}$ and $x' = 1$.

Given an AODE $F(y) = 0$ of order n, then $F(y) \in \mathbb{K}(x)\{y\}$.

Lemma 1. For each $k \geq 1$, there exists $R_k \in \mathbb{K}(x)\{y\}$ of order $n + k - 1$ such that

$$F^{(k)} = S_F \cdot y^{(n+k)} + R_k,$$

where $S_F := \frac{\partial F}{\partial y^{(n)}}$ is the separant of F.

Lemma 2. For $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$, we denote the coeff of x^k in f as $[x^k]f$. Then $[x^k]f = [x^0]\left(\frac{1}{k!}f^{(k)}\right)$.

Formal power series solutions

Let $F(y) = 0$ be an AODE of order n.

Using Lemmas 1 and 2, we have

Prop 1. Assume that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$. Then:

(i) $[x^0]F(x, z, \ldots, z^{(n)}) = F(0, c_0, \ldots, c_n)$.

(ii) For each $k \geq 1$, $[x^k]F(x, z, \ldots, z^{(n)})$ is equal to

$$\frac{1}{k!}\left(S_F(0, c_0, \ldots, c_n)c_{n+k} + R_k(0, c_0, \ldots, c_{n+k-1})\right),$$

where R_k is specified in Lemma 1.

Formal power series solutions

Let $F(y) = 0$ be an AODE of order n.

Theorem 1. Let $(c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ s.t. $F(0, c_0, \ldots, c_n) = 0$ and $S_F(0, c_0, \ldots, c_n) \neq 0$, and for each $k \geq 1$, we set

$$c_{n+k} = -\frac{R_k(0, c_0, \ldots, c_{n+k-1})}{S_F(0, c_0, \ldots, c_n)}.$$

Then $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a formal power series sol of $F(y) = 0$.

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Then $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a formal power series sol of $F(y) = 0$.

Example 1 (Continued). Consider the Riccati equation:

$$F(y) = y' - 1 - y^2 = 0.$$

Since $S_F = 1$, its formal power series sols are in bijection with

$$\{(c_0, c_1) \in \mathbb{K}^2 \mid c_1 = 1 + c_0^2\}.$$

Laurent series solutions

Let $z = \sum_{i=-r}^{\infty} c_i x^i \in \mathbb{K}((x))$. We call c_{-r} the lowest coeff of z, and denote it by $c(z)$.

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For $I = (i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1}$ and $r \in \{0, \ldots, n\}$, set $||I||_r := i_r + \ldots + i_n$. Write $||I||_0 = ||I||$. Moreover, set $||I||_\infty := i_1 + 2i_2 + \ldots + ni_n$.

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Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I(x) y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n} \in \mathbb{K}(x)\{y\}$ be of order n. Set:

$$\mathcal{E}(F) := \{I \in \mathbb{N}^{n+1} \mid f_I \neq 0\},$$
$$d(F) := \max\{||I|| \mid I \in \mathcal{E}(F)\},$$
$$\mathcal{D}(F) := \{I \in \mathcal{E}(F) \mid ||I|| = d(F)\}.$$

Laurent series solutions

Moreover, we denote

$$m(F) := \max\{\operatorname{ord}(f_I) + ||I||_\infty \mid I \in \mathcal{D}(F)\},$$
$$\mathcal{M}(F) := \{I \in \mathcal{D}(F) \mid \operatorname{ord}(f_I) + ||I||_\infty = m(F)\},$$
$$\mathcal{P}_F(t) := \sum_{I \in \mathcal{M}(F)} c(f_I) \cdot \prod_{r=0}^{n-1} (-t-r)^{||I||_{r+1}},$$

and if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, set

$$b(F) := \max\left\{ \frac{\operatorname{ord}(f_I) + ||I||_\infty - m(F)}{d(F) - ||I||} \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\}.$$

Laurent series solutions

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Definition 1. We call $\mathcal{P}_F$ the indicial polynomial of F at the origin.

Laurent series solutions

Theorem 2. (main result) Let $F(y) = 0$ be an AODE. If $r \geq 1$ is the order of a Laurent series sol of $F(y) = 0$ at the origin, then one of the following claims holds:

(i) $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, and $r \leq b(F)$;

(ii) r is an integer root of $\mathcal{P}_F(t)$.

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(i) $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, and $r \leq b(F)$;

(ii) r is an integer root of $\mathcal{P}_F(t)$.

- The proof is an analog of the Frobenius method for linear ODEs.
- Theorem 2 also holds for $x = \infty$.

Laurent series solutions

Example 2 (Continued). Consider:

$$F(y) = xy' + x^2y^2 + y - 1 = 0.$$

Assume $z \in \mathbb{K}((x))$ is a sol of $F(y) = 0$.

1. By Theorem 2, an order bound for the order of z is 2.
2. Substitute $z = \frac{1}{x^2}w$ with $w = \sum_{i=0}^{\infty} \frac{c_i}{i!}x^i \in \mathbb{K}[[x]]$ into $F(y) = 0$ and get a new AODE

$$G(w) = xw' + w^2 - w - w^2 = 0.$$

3. By Prop 1, we have

$$[x^0]G(w) = c_0^2 - c_0,$$
$$[x^k]G(w) = (2c_0 + k - 1)c_k + R_{k-1}(c_0, \ldots, c_{k-1}) \quad \text{for} \quad k \geq 1.$$

   Thus, $w = 1 + 0x + \frac{1}{3}x^2 + 0x^3 - \frac{1}{45}x^4 + \mathcal{O}(x^5)$.
4. Return $\frac{1}{x^2}w$.

Laurent series solutions

Theorem 2 gives a sharp order bound in Example 2. However, in general, it is not true.

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Example 3. Consider the linear ODE:

$$F(y) = x^2 y'' + 4xy' + (2+x)y = 0.$$

Assume $z \in \mathbb{K}((x))$ is a sol of $F(y) = 0$.

1. By Theorem 2, an order bound for the order of z is 2.
2. Substitute $z = \sum_{i=-2}^{\infty} c_i x^i \in \mathbb{K}((x))$ into $F(y) = 0$ and get
$$(1+i)(2+i)c_i + c_{i-1} = 0 \quad \text{for each} \quad i \in \mathbb{Z}. \tag{4}$$
Substitute $i = -1$ into (4) and get $c_{-2} = 0$. Thus, $F(y) = 0$ has no Laurent series sols of order 2.
3. Assume $c_{-1} = 1$. By (4), we conclude that $F(y) = 0$ has a sol of the form:
$$\sum_{i=-1}^{\infty} (-1)^{i+1} \frac{x^i}{(1+i)!(2+i)!}.$$

Laurent series solutions

Let $F(y) = 0$ be an AODE.

If $\mathcal{E}(F) = \mathcal{D}(F)$ and $\mathcal{P}_F(t) = 0$, then Theorem 2 gives no info for order bound of Laurent series sol of $F(y) = 0$.

Laurent series solutions

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Example 4. Consider the AODE:

$$F(y) = xyy'' - xy'^2 + yy' = 0.$$

Here, $\mathcal{E}(F) = \mathcal{D}(F)$ and $\mathcal{P}_F(t) = 0$. It has Laurent series sols

$$z = cx^{-n} \quad \text{for each} \quad c \in \mathbb{K} \ \text{ and } \ n \in \mathbb{N}.$$

Polynomial solutions

Let $F(y) = 0$ be an AODE, and $\mathcal{P}_{\infty,F}(t)$ be the indicial polynomial of $F(y) = 0$ at infinity.

Definition 2. We call $F(y) = 0$ noncritical if $\mathcal{P}_{\infty,F}(t) \neq 0$.

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By Theorem 2, if $F(y) = 0$ is noncritical, then there exists a bound for the degree of its polynomial sols.

Algorithm 1. Given a noncritical AODE $F(y) = 0$, compute all its polynomial sols.

1. Assume $z \in \mathbb{K}[x]$ is polynomial sol of $F(y) = 0$. Compute a degree bound B for z by Theorem 2.
2. Set $z = \sum_{i=0}^{B} c_i x^B$, where c_i is unknown. Substitute z into $F(y) = 0$ and solve the algebraic equations by using Gröbner bases. Return the results.

Polynomial solutions

Example 5 (Kamke 6.234). Consider:

$$F(y) = a^2 y^2 y''^2 - 2a^2 y y'^2 y'' + a^2 y'^4 - b^2 y''^2 - y'^2 = 0,$$

where $a, b \in \mathbb{K}$ and $a \neq 0$. Here, $\mathcal{P}_{\infty,F}(t) = a^2 t^2 \neq 0$.

Polynomial solutions

Example 5 (Kamke 6.234). Consider:

$$F(y) = a^2y^2y''^2 - 2a^2yy'^2y'' + a^2y'^4 - b^2y''^2 - y'^2 = 0,$$

where $a, b \in \mathbb{K}$ and $a \neq 0$. Here, $\mathcal{P}_{\infty,F}(t) = a^2t^2 \neq 0$.

1. Assume $z \in \mathbb{K}[x]$ is polynomial sol of $F(y) = 0$. By Theorem 2, a degree bound for z is 1.

2. Set $z = c_0 + c_1x$, where c_i is unknown. Substitute z into $F(y) = 0$ and solve the algebraic equations by using Gröbner bases. We find c, $c + \frac{x}{a}$, and $c - \frac{x}{a}$ are sols, where $c \in \mathbb{K}$.

Polynomial solutions

Example 4 (Continued). Consider the AODE:

$$F(y) = xyy'' - xy'^2 + yy' = 0.$$

Here, $\mathcal{E}(F) = \mathcal{D}(F)$ and $\mathcal{P}_{\infty,F}(t) = 0$. It has polynomial sols

$$z = cx^n \quad \text{for each} \quad c \in \mathbb{K} \ \text{ and } \ n \in \mathbb{N}.$$

Polynomial solutions

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- linear, first-order, quasi-linear second-order AODEs are noncritical.
- In Kamke's collection, all of the 834 AODEs are noncritical.

Rational function solutions

Consider a linear ODE:

$$F(y) = \ell_n y^{(n)} + \ell_{n-1} y^{(n-1)} + \cdots + \ell_0 y = 0,$$

where $\ell_i \in \mathbb{K}[x]$. The roots of ℓ_n are singularities of $F(y) = 0$.

Fact: Poles of rational sols of $F(y) = 0$ must be roots of ℓ_n.

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Fact: Poles of rational sols of $F(y) = 0$ must be roots of ℓ_n.

This is not true for nonlinear AODEs.

Example 6. Consider

$$F(y) = y' + y^2 = 0.$$

It has rational sols $z = \frac{1}{x-c}$ for $c \in \mathbb{K}$.

Rational function solutions

Question: Find a class of (nonlinear) AODEs s.t. the set of poles of rational sols of them is finite and computable.

Rational function solutions

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For $I, J \in \mathbb{N}^{n+1}$, we say $I \gg J$ if $||I|| \geq ||J||$ and $||I|| + ||I||_\infty > ||J|| + ||J||_\infty$.

For $I, J \in \mathbb{N}^{n+1}$, we say I and J are comparable if $I \gg J$ or $J \gg I$.

Given $S \subset \mathbb{N}^{n+1}$, we call $I \in S$ greatest element of S if $I \gg J$ for each $J \in S \setminus \{I\}$.

Definition 3. An AODE $F(y) = 0$ is called maximally comparable if $\mathcal{E}(F)$ admits a greatest element w.r.t. $\gg$.

Rational function solutions

Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I y^{i_0} (y')^{i_1} \ldots (y^{(n)})^{i_n} = 0$ be an AODE.

Theorem 3. Let $F(y) = 0$ be maximally comparable and I_0 be the greatest element of $\mathcal{E}(F)$ w.r.t. $\gg$. Then the poles of rational sols of $F(y) = 0$ are the zeros of $f_{I_0}(x)$ or infinity.

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- In Kamke's collection, 78.54% of the 834 AODEs are maximally comparable.

Rational function solutions

Algorithm 2. Given a maximally comparable AODE $F(y) = 0$, compute all its rational sols.

1. Compute the greatest element I_0 of $\mathcal{E}(F)$ w.r.t. $\gg$. Compute distinct roots $x_1, \ldots, x_m$ of $f_{I_0}(x)$.

2. Compute order bounds r_i and N for Laurent series sols of $F(y) = 0$ at x_i and infinity by Theorem 2, where $i = 1, \ldots, m$.

3. Set
$$z = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x - x_i)^j} + \sum_{k=0}^{N} c_i x^i,$$
where c_{ij}, c_i are unknown. Substitute z into $F(y) = 0$ and solve the algebraic equations by Gröbner bases.

Rational function solutions

Example 7. Consider the AODE:

$$\begin{aligned} F(y) &= x^2(x-1)^2 y''^2 + 4x^2(x-1)y'y'' - 4x(x-1)yy'' + \\ &\quad 4x^2 y'^2 - 8xyy' + 4y^2 - 2(x-1)y'' \\ &= 0. \end{aligned}$$

1. The greatest element of $\mathcal{E}(F)$ w.r.t. $\gg$ is $(0,0,2)$. By Theorem 3, the poles of rational sols of $F(y)=0$ might be $0, 1$ or infinity.
2. By Theorem 2, the order bounds of Laurent series sols of $F(y)=0$ at $0, 1$ and infinity are $0, 1$ and 1.
3. Set

$$z = \frac{c_1}{x-1} + c_2 + c_3 x \quad \text{for some} \quad c_1, c_2, c_3 \in \mathbb{K}.$$

Substitute z into $F(y)=0$ and we find $c_3 x$ and $\frac{1}{x-1} + c_3 x$ are rational sols of $F(y)=0$, where $c_3 \in \mathbb{K}$.

Conclusion

Let $F(y) = 0$ be an AODE of order n.

- Construct an order bound for Laurent series sols of $F(y) = 0$ and use it to compute them.
- An algorithm for computing polynomial sols of noncritical AODEs.
- An algorithm for computing rational sols of maximally comparable AODEs.

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Thanks!