Mahler Discrete Residues and Summability for Rational Functions

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Joint work with Carlos E. Arreche

Linear Mahler equations

Let $\mathbb{K}$ be an algebraically closed field of char 0, x be an indeterminate, and $p \in \mathbb{Z}_{\geq 2}$.

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \cdots + \ell_0(x)y(x) = f(x), \qquad (1)$$

where $\ell_i, f \in \mathbb{K}[x]$ are given, $y(x)$ is unknown. A solution of (1) is called a Mahler function.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

Fact: the generating series of any p-automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

Differential Galois Theory

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

Example (Roques 2018): A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over $\bar{\mathbb{Q}}(x)$.

Goal: Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$-field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^\times$ not root of unity and $\delta = x\frac{d}{dx}$.

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Let $z_1, \ldots, z_n \in F$, a $\sigma\delta$-extension of $\mathbb{K}(x)$ with $F^\sigma = \mathbb{K}$, satisfying

$$\sigma(z_i) = a_i z_i \qquad \text{for some} \quad a_1, \ldots, a_n \in \mathbb{K}(x)^\times.$$

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Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

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(Arreche 2017, Arreche-Z. 2022): Using (q-)discrete residues, there exist constants $m_1, \ldots, m_n \in \mathbb{K}$, not all 0, such that

$$m_1\frac{\delta(a_1)}{a_1} + \cdots + m_n\frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some $g \in \mathbb{K}(x)$ and $c \in \mathbb{K}$ (with $c = 0$ in case (S)).

Motivation

Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

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It also holds for the Mahler case. Question: How to derive the explicit formulae for $\mathcal{L}_1, \ldots, \mathcal{L}_n$?

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Idea: Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

Continuous residues

Let $\mathbb{K}$ be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_\alpha(k) \in \mathbb{K}$ (almost all 0).

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Then $f(x)$ is rationally integrable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $g'(x) = f(x)$, if and only if the (continuous first-order) *residues*

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Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability* $\mapsto$ *summability*) for the *shift* ($x \mapsto x+1$) and *q-dilation* ($x \mapsto qx$) difference operators.

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k,n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_\alpha(k,n) \in \mathbb{K}$.

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The discrete residue of $f(x) \in \mathbb{K}(x)$ at the $\mathbb{Z}$-orbit $[\alpha] \in \mathbb{K}/\mathbb{Z}$ of order k is defined as

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Proposition (Chen-Singer 2012) $f(x)$ is rationally summable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(x+1) - g(x)$, if and only if $\mathrm{dres}(f, [\alpha], k) = 0$ for each $[\alpha] \in \mathbb{K}/\mathbb{Z}$ and $k \in \mathbb{N}$.

Mahler summability for rational functions

Fix $p \in \mathbb{Z}_{\geq 2}$ and let the Mahler difference operator $\sigma : g(x) \mapsto g(x^p)$ for $g(x) \in \mathbb{K}(x)$.

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Mahler Summability Problem: given $f(x) \in \mathbb{K}(x)$, decide effectively whether $f(x)$ is Mahler summable.

- Done by Chyzak-Dreyfus-Dumas-Mezzarobba (2018).

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Our Goal: Construct a ($\mathbb{K}$-linear) complete obstruction to the Mahler summability of $f(x) \in \mathbb{K}(x)$.

Mahler summability for rational functions

More precisely, for the $\mathbb{K}$-linear map $\Delta : g(x) \mapsto g(x^p) - g(x)$, we wish to construct explicitly a $\mathbb{K}$-linear map ∇ on $\mathbb{K}(x)$ such that $\mathrm{im}(\Delta) = \ker(\nabla)$, bypassing computation of certificates.

We call ∇ the Mahler reduction operator. Given $f \in \mathbb{K}(x)$, set $\bar{f} = \nabla(f)$. Then f is Mahler summable if and only if $\bar{f} = 0$. The numerators in the partial fraction decomposition of $\bar{f}$ are Mahler discrete residues of f.

Mahler trajectories and Mahler trees

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We denote by $\mathbb{Z}/\mathcal{P}$ the set of maximal trajectories for the action of $\mathcal{P}$ on $\mathbb{Z}$ by multiplication:

$$\mathbb{Z}/\mathcal{P} = \Big\{\{0\}\Big\} \cup \Big\{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \;\Big|\; i \in \mathbb{Z} \text{ such that } p \nmid i\Big\}.$$

The elements $\theta \in \mathbb{Z}/\mathcal{P}$ are pairwise disjoint subsets of $\mathbb{Z}$ whose union is all of $\mathbb{Z}$.

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We denote by $\mathcal{T}_M$ the set of equivalence classes for the equivalence relation on $\mathbb{K}^\times$ defined by $\alpha \sim \gamma$ if and only if $\alpha^{p^s} = \gamma^{p^r}$ for some $r, s \in \mathbb{Z}_{\geq 0}$.

The elements $\tau \in \mathcal{T}_M$, called Mahler trees, are pairwise disjoint subsets of $\mathbb{K}^\times$ whose union is all of $\mathbb{K}^\times$.

Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

$$f_L(x) := \sum_{j\in\mathbb{Z}} r_j x^j \qquad \text{and} \qquad f_T(x) := \sum_{k\geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

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Moreover, the decompositions $f_L = \sum_{\theta \in \mathbb{Z}/\mathcal{P}} f_\theta$ and $f_T = \sum_{\tau \in \mathcal{T}_M} f_\tau$:

$$f_\theta := \sum_{j \in \theta} r_j x^j \qquad \text{and} \qquad f_\tau := \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{c_\alpha(k)}{(x-\alpha)^k}$$

are also σ-stable. Can decide summability of f by deciding for each f_θ ($\theta \in \mathbb{Z}/\mathcal{P}$) and each f_τ ($\tau \in \mathcal{T}_M$) individually.

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j\in\mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of $f(x)$ at infinity is the vector

$$\mathrm{dres}(f, \infty) := \left(\sum_{j\in\theta} r_j\right)_{\theta\in\mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta\in\mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

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- The definition (and proofs) for Mahler discrete residues at Mahler trees is similar in spirit, but more technical.

Main Result

Theorem (Arreche-Z. 2022) Given $f \in \mathbb{K}(x)$. Then f is Mahler summable if and only if $\mathrm{dres}(f, \infty) = \mathbf{0}$ and $\mathrm{dres}(f, \tau, k) = \mathbf{0}$ for all $k \in \mathbb{N}$ and $\tau \in \mathcal{T}_M$.

Thanks!